Symbols for toric Eisenstein cocycles and arithmetic applications

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ABSTRACT. Using a complex parameterizing rational spherical chains, we construct explicit cocycles for $\mathrm{GL}_n(\mathbb{Q})$ valued in the motivic cohomology of (open subsets of) the algebraic n-torus $\mathbb{G}_m^n$. The resulting cocycles directly generalize the work of Sharifi and Venkatesh from the case $n = 2$. Even in this special case, our systematic use of pushforwards allows us to avoid the use of their "connecting sequences," and allows us to refine the construction and Hecke properties of Sharifi's map ϖ to the maximal expected statements, while inverting only the prime 2. For general n, the $d\log$ regulator of our cocycle is related by convex conical duality to cocycles constructed from Shintani cones. This affords a systematic approach to p-adic L-functions for totally real fields without need for auxiliary data or logarithm sheaf coefficients, including a distribution-valued $\mathrm{GL}_n(\mathbb{Z})$-cocycle specializing in a simple way to all such p-adic L-functions. It moreover provides a direct conceptual link between polylogarithmic constructions of Eisenstein classes (e.g., Beilinson–Kings–Levin), and those constructed using Shintani cones (e.g., Charollois–Dasgupta–Greenberg or Dasgupta–Spiess). We also show how our formalism gives an alternate, purely algebraic proof of the exceptional divisibilities of the Deligne-Ribet 2-adic L-function (in almost all cases).

CONTENTS

1. Introduction

The degree-n Milnor K-theory of a field F, denoted $K_n^M(F)$, is a quotient of $F^\times \otimes \ldots \otimes F^\times$ by certain ("Steinberg") relations, with a general element written $\{f_1, \ldots, f_n\}$ for $f_1, \ldots, f_n \in F^\times$. When $F = \mathbb{Q}(z_1, \ldots, z_n)$, which we think of as the fraction field of the n-fold power of the multiplicative group $\mathbb{G}_m^n$, the simple-looking elements given by GL_n-translates of

$$\{1 - z_1, \ldots, 1 - z_n\},$$

contain surprisingly interesting arithmetic information: for $n = 1$, specializing them at Nth-roots of unity leads to N-cyclotomic units, and when $n = 2$, similar specializations are also important in cyclotomic Iwasawa theory, and in particular the Sharifi conjectures. For $n > 2$, the specializations in K-theory are not as interesting, but one can formally take the coordinate-wise logarithmic derivative of these Milnor K-theory elements to get generating series which look like

$$\frac{e^{2\pi i(t_1+\ldots+t_n)}}{(1-e^{2\pi i t_1})\ldots(1-e^{2\pi i t_n})}$$

where $z_i = e^{2\pi i t_i}$, $1 \leqslant i \leqslant n$, which are generating series of generalized Bernoulli numbers. For $n = 1$, the Mellin transform of this series is used classically in the integral representation of the Riemann zeta function

$$\xi(s) := \Gamma(s/2)\zeta(s) = \int_0^\infty \frac{x^s}{e^x - 1}\frac{dx}{x}$$

and thereby to prove the functional equation and find rational formulas for its values at negative integers; in the general case, this method was generalized by Shintani to to L-values of totally real fields of degree n [Shin], using a formalism which viewed these series as generating series associated to rational polyhedral cones in $\mathbb{R}^n$ (which is often called the "Shintani method").

In [SV], with applications to the Sharifi conjectures in mind, the authors construct 1-cocycles for $\mathrm{GL}_2(\mathbb{Z})$ and its congruence subgroups, taking values in elements like $\{1 - z_1, 1 - z_2\}$ in the second Milnor K-group $K_2^M(k(\mathbb{G}_m^2))$. In this article, we generalize their approach by constructing analogous cocycles for (subgroups of) $\mathrm{GL}_n(\mathbb{Q})$ valued in $K_n^M(k(\mathbb{G}_m^n))$, or various refinements/specializations thereof. When $n = 2$, our formalism enables us to to improve on their results and obtain new integrality and Hecke equivariance properties for one of the maps in the Sharifi conjectures, while at the same time being more geometrically-flavored and requiring fewer technical computations.

The $d\log$ regulator of our cocycle yields a new construction of an "Eisenstein cocycle" containing a large amount of interesting arithmetic information, closely related to many previous cocycles appearing in the literature. In particular, our cocycles bear a "conical dual" relation to cocycles constructed based on Shintani's techniques, which we will exploit to give a systematic treatment of the integrality properties of L-values for totally real fields, including a conceptual proof of "exceptional" 2-adic divisibilities previously only known by explicit automorphic computations.

1.1. **Relation to existing work.** The primary inspiration for the methods of this article was the motivic $\mathrm{GL}_2(\mathbb{Z})$-Eisenstein cocycle defined by [SV]: our main symbol construction can in large part be seen as an extension of the method of [SV, §5] from $n = 2$

to general n, though they use toric geometry in place of Bloch cycles to prove relations, and do not work with pushforwards. In the arithmetically interesting case $n = 2$, we are able to refine their result by computing that the Hecke action is the expected one on a certain modular symbol, rather than just a cohomology class. In particular, we use a new method to obtain a Hecke-equivariant map on homology of the modular curve relative to cusps, rather than on the closed modular curve. Using our tools, we are also able to directly compute the Hecke actions at primes dividing the level while only inverting the prime 2, improving on both Sharifi-Venkatesh's results as well as the work of Lecouturier-Wang [LcW] on operators dividing the level.

Also closely related to the present work is the approach of Shintani [Shin] using rational generating functions associated to rational polyhedral cones, originally formulated in a non-cohomological way to compute L-values of totally real fields. Eisenstein cocycles built on this approach have previously appeared in the literature, e.g. in [SH], [CD] and [CDG], [Hill], and [LP], but our approach is more systematic: besides being equally applicable to the motivic setting, we are able to quotient by very few relations in the coefficients, and do not need auxiliary data like lexicographic orderings (as in Richard Hill's approach [Hill]) or perturbations (as in the cocycle of Charollois-Dasgupta-Greenberg [CDG], after an idea of Colmez); this extra data implicitly appears only in the "degenerate" values of a choice of cocycle representative.

We achieve this by working with a symbol complex of *spherical chains*, whose realizations are related to *pushforwards* of generating functions of the *dual* cones to those considered by Shintani. In fact, our symbol complex construction is more directly tied to "polylogarithmic" constructions of Eisenstein classes, as in the work of Beilinson-Kings-Levin [BKL] as well as Bergeron-Charollois-Garcia [BCG1] [BCG2], a term we use to refer to Eisenstein classes constructed by specifying residues along torsion cycles in some algebraic or topological group. We make this relationship precise in Appendix A. The terminology "polylogarithmic" reflects that the more general versions of these cocycles (as in [BKL]) use coefficients in a large "logarithm" sheaf in order to obtain a direct relation to higher-weight specializations of Eisenstein series, which is necessary to prove the higher-weight interpolation for the corresponding constructions of p-adic L-functions. (We also note that [BHYY] relatedly constructs a equivariant polylogarithm class in *Deligne* cohomology of the torus corresponding to a totally real field, which as noted in Remark 3.3, should be the restriction of the Deligne regulator of our motivic class $\Theta(n)$ upon projection to constant coefficients.)

These polylogarithmic-type constructions generally are closely linked to actual analytic Eisenstein series representatives, while the connection between Shintani-style constructions and actual Eisenstein series is largely more indirect. For example, Cassou-Noguès' [CN] approach to p-adic L-functions using Shintani's method produces the same objects as the approach of Deligne-Ribet [DR] using Hilbert-Eisenstein families, since they interpolate the same L-values, but the conceptual link between the two approaches is not clarified by this. By producing a cocycle which is both polylogarithmic (from Appendix A) and related by conical duality to Shintani generating series, we bridge this gap. The relation to Shintani's method also allows us to prove the interpolation property of our p-adic L-functions at higher weights using only trivial coefficients, rather than logarithm sheaf.

We also bridge a gap between the p-adic congruences obtained by [DR] and [CN]: the former obtains extra 2-adic divisibilities coming from a computation of the constant term of Hilbert-Eisenstein series. The issue of whether these could be "seen" by a Shintani cone approach was already raised in [DR]; this was also considered by Gross [Gro], who put it into an equivariant algebraic framework, and very recently by Colmez [Col], who obtained partial progress (see Théorème 3.22, Remarque 3.23 in loc. cit). Our algebraic setup leads to a proof of the full 2-adic divisibility, except in the particularly delicate case of unramified totally odd characters at weight $s = 0$; see Theorem 1.4.

Finally, to review the relation of our cocycle with a few others not yet mentioned: in the motivic setting, Lim and Park [LP] based off an idea of Stevens [Ste2], also construct a cocycle valued in Milnor K-theory of a ring of "trigonometric functions". Their construction is a complex-analytic and "infinite level", with their ring of trigonometric functions corresponding to the functions on a pro-tower of algebraic tori over all finite isogenies; however, like other previous Shintani-style approaches [CDG] and [Hill], their construction is "conical dual" to ours, and involves addditional quotients by relations. The duality relation they exhibit in [LP, §3], between the regulator of their motivic cocycle and a naive version of the Shintani cocycle of [Hill], inspired our understanding of the conical duality linking our construction to previous ones in the literature.

1.2. **Structure of the paper.** We briefly the structure of the paper: the preliminaries consist of a review of needed facts on motivic cohomology in Section 2.1, followed by the construction of the symbol complex and resulting cocycles in Sections 2.2 and 2.3. Section 3, on the motivic cocycle and applications, and Section 4, on the de Rham cocycle and applications, are mostly independent; readers interested in one or the other may safely skip around and refer back to the preliminary sections as needed. The only major exception is that existence of the de Rham cocycle is proven using Theorem 3.1 in the motivic section, since we construct the de Rham cocycle as the regulator of the motivic one.

1.3. **Summary of methods and results.** We now give more precise formulations of our results and arguments, to the extent possible without introducing excessive technical background.

Our main initial construction is:

Theorem 1.1. *There exists a Milnor K-theory-valued cohomology class*

$$\Theta(n) \in H^{n-1}(\mathrm{GL}_n(\mathbb{Q}), K_n^M(\mathbb{Q}(z_1^{\pm}, \dots, z_n^{\pm}))/\{-z_1, \dots, -z_n\})$$

with a homogeneous cocycle representative given by

$$(\gamma_1, \dots, \gamma_n) \mapsto \begin{pmatrix} \ell_1 & \dots & \ell_n \end{pmatrix}_* \{1 - z_1, \dots, 1 - z_n\}$$

for any $\gamma_1, \dots, \gamma_n$ whose respective first columns $(\ell_1, \dots, \ell_n)$ are linearly independent.

This theorem was previously proven for $n = 2$ in [SV]. Our method, which is a variant of the approach in [SV, §5], is to construct an explicit $\mathrm{GL}_n(\mathbb{Q})$-module of *symbols* $C(n)$ parameterizing elements in $K_n^M(\mathbb{Q}(z_1^{\pm}, \dots, z_n^{\pm}))/\{-z_1, \dots, -z_n\}$. We refer to these as symbols, as they are in an explicit combinatorial way and are easy to write down.

The module $C(n)$ is constructed as the top-degree chains of a chain complex computing the homology of the $(n-1)$-sphere, modulo the fundamental class, which we refer to as the symbol complex. The $\mathrm{GL}_n(\mathbb{Q})$-action on the resulting length-$(n+1)$ exact complex yields explicit $(n-1)$-cocycles via a lifting process, all representing the cohomology class. Readers interested only in arithmetic applications may safely skip the details, referring only when needed to the characterization of the symbol complex $\mathrm{Chains}(n)$ in Proposition 2.4.

Most of the work of constructing $\Theta(n)$ is in proving that the relations are satisfied in Milnor K-theory, which we prove by constructing explicit corresponding boundaries in a cohomology complex (the *Bloch cycle complex*) computing Milnor K-theory. The details of this realization are in section 3.1. Notably, our systematic use of *pushforwards* in the realization maps allows us both to extend our cocycle to $\mathrm{GL}_n(\mathbb{Q})$ (rather than just $\mathrm{GL}_n(\mathbb{Z})$), and to avoid the technicalities of the "connecting sequences" used in [SV], as we can work directly with non-unimodular symbols instead of needing to decompose them into unimodular ones. This also enables our simpler treatment of Hecke actions (see below).

As in [SV, §5], we also consider variants of the construction in this section, there are various methods by which one can remove the necessity of quotienting by $\{-z_1, \dots, -z_n\}$, (though usually one gives up something else; e.g. inverting primes). Conversely, by quotienting out by *more* relations, one can obtain an *parabolic* version of $\Theta(n)$. Our use of symbol complexes makes it possible to understand systematically what modifications are required for each of these variants.

1.3.1. *Application to Sharifi maps.* In particular, a modular symbol variant of the cocycle has the following application: in the case $n = 2$, Sharifi and Venkatesh use their parabolic cocycle to construct and prove the Eisenstein property of one of the maps which figures in the eponymous Sharifi conjectures, relating the homology of modular curves and algebraic K-theory of rings of integers in cyclotomic fields. Their methods produce the desired map in the form

$$\Pi_N : H_1(X_1(N), \mathbb{Z}) \to K_2(\mathbb{Z}[1/N, \zeta_N]) \otimes \mathbb{Z}[1/2]$$

which is Eisenstein for the anemic Hecke operators only, i.e. factors through the quotient by the operators $T_\ell - 1 - \ell\langle\ell\rangle$ for $\ell \nmid N$. The Sharifi conjectures concern an explicitly constructed map

$$\Pi_N^\circ : H_1(X_1(N), C_1^\circ(N), \mathbb{Z}) \to K_2(\mathbb{Z}[1/N, \zeta_N]) \otimes \mathbb{Z}[1/2], \quad [u : v] \mapsto \{1 - \zeta_N^u, 1 - \zeta_N^v\}$$

on the larger cohomology group relative to the set of cusps $C_1^\circ(N)$ not in the $\Gamma_0(N)$-orbit of ∞, and work of Fukaya-Kato [FK] showed that for $p|N$, the complex conjugation-fixed ("plus") version of this map, denoted by subscript $+$,

$$(\Pi_N^\circ)_+ \otimes \mathbb{Z}_p : H_1(X_1(N), C_1^\circ(N), \mathbb{Z}_p)_+ \to (K_2(\mathbb{Z}[1/N, \zeta_N]) \otimes \mathbb{Z}_p)_+$$

given by the formula

$$[u : v] \mapsto \{1 - \zeta_N^u, 1 - \zeta_N^v\}_+ := \frac{1}{2}(\{1 - \zeta_N^u, 1 - \zeta_N^v\} + \{1 - \zeta_N^{-u}, 1 - \zeta_N^{-v}\})$$

factored through not only the anemic Eisenstein operators $T_\ell - 1 - \ell\langle\ell\rangle$, but also $U_p - 1$ whenever $p|N$. Thus, Sharifi-Venkatesh's work improved on this by constructing an anemically Eisenstein map with $\mathbb{Z}[1/2]$-coefficients, and not only on the plus part. However, they were unable to compute the action of the Hecke operators dividing the level, or to define it on the larger relative cohomology group as in Π_N°. Later, [LcW] combined the methods of Sharifi-Venkatesh and Fukaya-Kato to show that the map $\Pi_N \otimes \mathbb{Z}[1/6]$ factors through the non-anemic Hecke operators $U_\ell - 1$ for $\ell|N$, again only on the plus part.

After "spreading out" the pullback of the modular symbol variant of our cocycle $\Theta^{MS}(2)$ in order to pull it back by an Nth root of unity, we are able to obtain the improved result:

Theorem 1.2. *The map*

$$(\Pi_N^\circ)_+ : H_1(X_1(N), C_1^\circ(N), \mathbb{Z}) \to K_2(\mathbb{Z}[1/N, \zeta_N]) \otimes \mathbb{Z}[1/2]$$

factors through both $T_p^* - p - \langle p\rangle$ for $p \nmid N$ and $U_p - 1$ for $p|N$. Further, the diamond operator $\langle d\rangle$ acts as pullback by the Galois automorphism $[d] : \zeta_N \mapsto \zeta_N^d$ for all $d \in (\mathbb{Z}/N)^\times$.

Additionally, if N is a power of a prime, or if we further restrict the set of cusps to exclude $\Gamma_0(p)$-orbit of ∞ for all *$p|N$, then all of this holds even for the map Π_N°,* without *projecting to the $+$ part.*

The point is that these relative homology groups are naturally $\Gamma_1(N)$-subquotients of the Steinberg module $\mathrm{St}(2)$, consisting of mass-zero finite functions on $\mathbb{P}^1(\mathbb{Q})$; this, in turn, is a $\mathrm{GL}_2(\mathbb{Q})$-equivariant quotient of our spherical chains module $C(n)$. Our realization map allows us to understand precisely what elements in K-theory we need to quotient by to have these extra Steinberg relations, resulting in a $\mathrm{GL}_n(\mathbb{Q})$-equivariant modular symbol

$$\Theta^{MS}(2) : \mathrm{St}(2) \to K_2(k(\mathbb{G}_m^2))/\text{extra relations}$$

and which pulls back, upon restriction to $\Gamma_1(N)$-level, to a modular symbol

$$\mathrm{St}(2)_{\Gamma_1(N)} \to K_2(\mathbb{Z}[1/N, \zeta_N])/\text{extra relations} \otimes \mathbb{Z}[\tfrac{1}{2}]$$

where these extra relations are trivial with $\mathbb{Z}[\frac{1}{2}]$-coefficients upon projection to the plus part (or are already zero, in the special case noted in the theorem).

Moreover, the Hecke action can be computed directly on the modular symbol $\Theta^{MS}(2)$ (without ever passing to a cohomology class), in terms of a simple geometric operation on the torsion sections used to pullback. This enables us to compute the Hecke action at all primes; we find the Eisenstein property holds if one restricts the set of cusps as indicated in the theorem. We further give a geometric interpretation and proof of the N-integrality of a restriction of Π_N° to the homology of the closed curve $X_1(N)$, which previously had been proven computationally [SV, Lemma 4.2.7] [FK, Lemma 3.3.11].

1.3.2. *Application to L-values.* We now turn to the *regulator* of $\Theta(n)$, i.e. its composition with the map

$$\{f_1, \dots, f_n\} \mapsto d\log f_1 \wedge \dots \wedge d\log f_n \in \Omega^n_{k(\mathbb{G}_m^n)}.$$

After contracting with the $\mathrm{SL}_n(\mathbb{Q})$-invariant vector $\partial z_1 \wedge \dots \wedge \partial z_n$, the resulting de Rham cocycle $\Theta^{dR}(n)$ is valued in precisely the same kind of generating functions associated by Shintani [Shin] to arrangements of cones, closely related to L-functions of totally real fields. However, the generating function associated to the class of a rational simplex $\Delta \subset S^{n-1}$ on the $(n-1)$-sphere is *not* the generating function of the associated (open) cone, but to its *dual* cone: that is, formally, to the sum

$$(t_1, \dots, t_n) \mapsto \sum_{\substack{x \in \mathbb{Z}^n \\ \langle x, \mathbb{R}_{\geqslant 0}\Delta\rangle > 0}} e^{2\pi i \langle x, t\rangle}$$

rather than

$$(t_1, \dots, t_n) \mapsto \sum_{x \in \mathbb{Z}^n \cap \mathbb{R}_{>0}\Delta} e^{2\pi i \langle x, t\rangle},$$

where $\langle -, - \rangle$ is the usual inner product, and we identify a region of S^{n-1} with the space of rays in its $\mathbb{R}^+$-span. In [CDG], for example, the Shintani generating function associated to the anisotropic torus associated to a totally real field F of degree n is of the latter form (and also involves delicate choices of lower-dimensional cones), while the specialization of our $\Theta^{dR}(n)|_{\mathcal{O}_F^\times}$ is of the former. However, we are able to show that L-value specializations of the latter type of series are the same as those coming from ours $\Theta^{dR}|_{\mathcal{O}_F^\times}$, if one takes the *inverse transpose* of the embedding $\mathcal{O}_F^\times \hookrightarrow \mathrm{GL}_n(\mathbb{Z})$: this reflects the underlying conical duality, and involves a technical argument using stabilized versions of our symbol complex at an auxiliary prime, as well as a geometric incarnation of conical duality in the Tits building of GL_n (see Appendix B).

As a result of the smoothing, we are able to pull back $\Theta^{dR}(n)$ along *all* torsion sections, and hence obtain a single class valued in distributions which we can show specializes to all different p-adic L-functions associated to totally real fields of degree n:

Theorem 1.3 (Theorem 4.16)**.** *For each integer $c > 1$ prime to a prime p, there exists a cohomology class valued in a certain space of p-adic distributions (see Section 4.1.2 and the formula* (4.11) *for notational details)*

$${}_c\Theta^{dR}_{\mathbf{D}_p}(n) \in H^{n-1}(\mathrm{GL}_n(\mathbb{Z}), \mathbf{D}^{(0)}((\mathbb{Q}_p^n)^\vee, \mathbb{Z}_p)(\mathrm{sgn})^{(0)}),$$

such that if F is a totally real field with $[F:\mathbb{Q}] = n$ and $\alpha : I \xrightarrow{\sim} (\mathbb{Z}^n)^\vee$ is a choice of dual basis for a fractional ideal $I \subset F$, then the restriction of ${}_c\Theta^{dR}_{\mathbf{D}_p}(n)$ to the resulting embedded copy of $U^+ := (\mathcal{O}_F^\times)^+ \hookrightarrow \mathrm{GL}_n(\mathbb{Z})$ satisfies

$$(-1)^{n-1} \int_{(\mathbb{Q}_p^n)^\vee} \psi(\alpha^{-1}(t)) \mathbf{N}_{\mathbb{Q}}^F(\alpha^{-1}(t))^k \, d({}_c\Theta^{dR}_{\mathbf{D}_p}(n)|_{U^+} \frown c_{U^+}) = \zeta_I(\psi, -k) - c^n \zeta_{(c)I}(\psi, -k)$$

for any function $\psi : I^+/U^+ \to \overline{\mathbb{Q}_p}$ locally constant in the p-adic topology.[1] *Here c_{U^+} is a fundamental class of $U^+ \cong \mathbb{Z}^{n-1}$, positively oriented in an appropriate sense, and*

$$\zeta_I(\psi, -k) := \left(\sum_{\lambda \in I^+/U^+} \psi(\lambda) \mathbf{N}\lambda^{-s} \right)_{s=-k}$$

is a partial zeta value for I.

The fact that we use Shintani's method (in Section 4.2 onward) to prove the p-adic interpolation is of note, when juxtaposed with our "polylogarithmic"-style construction (see Appendix A) of our classes. Previously, [BKL] constructed p-adic L-functions for totally real fields using Eisenstein classes constructed polylogarithmically on topological tori $(S^1)^n$, but needed coefficients in the logarithm sheaf to prove the interpolation property at higher weights; the direct relationship with Shintani's method allows us to avoid this.

Finally, we remark that we can also obtain S-arithmetic or even $\mathrm{GL}_n(\mathbb{Q})$ versions of this class, and representing cocycles (maybe after inverting primes in the coefficients). We do not go into S-arithmetic applications in this article, so this is only covered in passing; however, for example, we expect it to give an alternative approach to the rigid analytic cocycles of [RX2].

The construction of p-adic L-functions by Deligne-Ribet [DR] also proved extra divisibilities at the prime 2 for certain L-values "with parity"; it remained an open question whether approaches using Shintani cones (like [CN], contemporaneously) could "see" these extra congruences; for example, this was considered by Gross [Gro, Proposition 5.4] (who formulated it in algebraic terms quite related to our distributions) and Colmez [Col, Théorème 3.22] (who proved a slightly weaker statement). We prove the full expected congruence for the values we are able to interpolate:

Theorem 1.4 (Theorem 4.23, analogue of (8.11–12) in [DR])**.** *Let $\psi : G_{2^\infty} \to \overline{\mathbb{Q}_2}$ be a totally odd continuous function on the 2^∞-narrow ray class group of F. Then, for our 2-adic zeta element ${}_{\mathfrak{c}}\zeta_2^F$ constructed in Section 4.3.4, we have the congruence*

$$\int_{G_{p^\infty}} \psi(J)\, d_{\mathfrak{c}}\zeta_2^F(J) \equiv 0 \pmod{2^n}.$$

This theorem follows in a very formal way from our setup; as such, by comparison with the method of [DR], one could view it as a "cohomological explanation" for the appearance of 2^n in the constant term of Hilbert–Eisenstein series. As in [DR], we in fact prove the theorem for more general functions on a monoid of ideals strictly larger than the p^∞-ray class group. In this more general setting, the congruence is slightly more subtle and is only modulo 2^{n-1} in some cases, requiring some more delicate considerations in class field theory. We are able to prove this more delicate statement using our purely algebraic methods, except for in some particularly subtle cases: see Theorem 4.23 for the precise statement and more discussion of this.

Note that essentially identical results about L-values hold if one introduces ramification at some tame level N prime to p. Since these results were already proven by [DR] and the main innovation of this paper is to exhibit "purely cohomological" reasons for these congruences, we decided to focus on the case with no tame level to present the idea more simply. However, all our methods generalize to the case of tame level, essentially without change.

1.4. **Future work.** The basic methods (parameterization by linear algebraic complexes of symbols) of this article apply, though with some substantial changes, to the setting of elliptic polylogarithms: for example, the action of $\mathrm{GL}_n/\mathbb{Q}$ on the nth power of the universal elliptic curve, or GL_n/K on the nth power of an elliptic curve with CM by K. This elliptic case will be the subject of

[1]For reasons of brevity, we do not consider adding tame level - i.e., prime-to-p level - to our p-adic L-functions; however, the same constructions all work in this generality as well.

a sequel to this article. Working analytically over $\mathbb{C}$, certain specializations of the former setting yield (after pullback by torsion sections) the Siegel unit-valued modular symbols constructed in the recent work of [BPPS]. This elliptic case has analogous applications to the arithmetic of totally real fields, as well as to the Sharifi conjectures for the pairs $(\mathrm{GL}_3(\mathbb{Q}), \mathrm{GL}_2(\mathbb{Q}))$ and $(\mathrm{GL}_2(K), \mathrm{GL}_1(K))$ in the case of a CM field K, just as the present article deals with the case $(\mathrm{GL}_2(\mathbb{Q}), \mathrm{GL}_1(\mathbb{Q}))$. In the setting of Drinfeld modules for a function field $F/\mathbb{F}_q$, we also use symbol complex methods to attack the case $(\mathrm{GL}_n(F), \mathrm{GL}_{n-1}(F))$, in forthcoming joint work with the the first author of [SV].

Another question posed by [Katz] is the relation between Shintani cones and toroidal compactifications of Hilbert-Blumenthal varieties; via Hirzebruch's conjecture (now a theorem), this also is closely related to the same L-values studied by Deligne-Ribet (as well as the present article). We would be interested in a group theoretic/geometric answer to this question, which seems related to our cohomological methods.

Finally, we do not go into the relation of our cocycles to Gross-Stark units in this article, but it may be of interest to write down the relationship carefully. As noted in Remark **??**, we believe that our "de Rham cocycle" results in the same (smoothed) distributions for totally real fields as one obtains from [CDG], for example, and thus should yield the same now-proven formulas for Gross-Stark units of [DS] (thanks to the work of [DKSW]). These other existing cocycles are already perfectly sufficient for explicit class field theory, but the very natural algebraic formulation of integrality properties in our approach could possibly offer benefits of exposition, at least.

We *do* write down the relationship with Gross-Stark units of a closely-related cocycle (from the polylogarithm of a *topological* torus) in the joint work [RX], as well as its relation to the rigid analytic cocycles of [DV]; in joint work in preparation, we will relate that polylogarithmic cocycle to the one in this article.

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2. Preliminaries

2.1. **Motivic cohomology.** In this section, we recall some needed facts about motivic cohomology and Milnor K-theory. For a smooth equidimensional scheme X over a discrete valuation ring (DVR) R, we will compute (or, for the purposes of this article, *define*) the motivic cohomology $H^i(X, \mathbb{Z}(n))$ as the cohomology $H^i(X, \mathbb{Z}(n)_X)$ of Bloch's weight-n *cubical* cycle complex $\mathbb{Z}(n)_X$; this is the approach followed in [Tot] and [GL], for example, over a field.

The Bloch cycle complex is defined as follows: let

$$\tilde{z}^n(X, i) := Z^n(U \times \square^i)$$

be the group of codimension-n cycles on $X \times \square^i$ meeting all faces properly. Here, $\square^i$ is the algebraic i-cube which we identify with $(\mathbb{P}^1 - \{1\})^n$ and the jth face map is given by the difference of the pullbacks to the subvarieties cut out by $t_j = 0$, respectively $t_j = \infty$; the alternating sum of face maps gives, as usual, a differential from $\tilde{z}^n(X, i)$ to $\tilde{z}^n(X, i-1)$.[2] It turns out that the resulting complex splits into a direct sum

$$\tilde{z}^n(X, i) = d^n(U, i) \oplus z^n(X, i)$$

[2] In some conventions, $\square^i$ is identified with $\mathbb{A}^i$, and the face maps are given by $t_j = 0, 1$ instead of $0, \infty$. This differs from our convention by a Möbius transformation; ours is the convention used by [Tot], and we we find it to be more convenient for later applications.

where the former summand consists of *degenerate* cycles which can be pulled back from one of the faces of $\square^i$ given by $t_j = 0$, and the latter summand consists of the *reduced cycles* which are in the kernel of of the restriction to each face $t_j = 0$. We define

$$(\mathbb{Z}(n)_X)^i := z^n(X, 2n - i).$$

This complex is suitably functorial for flat pullbacks and proper pushforwards.

More generally, over a Dedekind scheme base, we can define motivic cohomology as the *hypercohomology* of the Zariski sheafification of the Bloch complex: the fact that this coincides with our above (non-hypercohomology) definition for the Bloch cycle complex was proven for fields as [FS, Corollary 12.2], and for DVRs in [Lev2]; consult these sources for details. The only fact we need about this broader definition is the existence in this generality of a localization sequence:

Proposition 2.1 ([Geis]). *If $Z \subset X$ is a pair of smooth schemes embedded with pure codimension d, over a Dedekind domain, then there is a localization sequence*

$$\ldots \to H^{i-2d}(Z, \mathbb{Z}(n-d)) \to H^i(X, \mathbb{Z}(n)) \to H^i(X - Z, \mathbb{Z}(n)) \to H^{i+1-2d}(Z, \mathbb{Z}(n-d)) \to \ldots$$

This will be used to extend results about pullbacks of our motivic cocycles to bases larger than DVRs.

Remark 2.2. This article is not concerned with the minutiae of motivic constructions, so we only remark briefly on the technical details we are glossing over in the above definition, in order to provide justifying references: first, this construction should more properly be called Borel-Moore motivic homology; historically, Bloch called them "higher Chow groups". For smooth schemes over a field, higher Chow groups agree with the usual modern construction of motivic cohomology defined via Voevodsky-style motivic complexes, thanks to the results in [V] and [FS]. Second, simplicial language is more standard than cubical (as in, e.g., [Geis]), but the two approaches are equivalent for formal reasons, as proven in [GL] over a field (though the proof works also over a DVR).

When $X = \operatorname{Spec} F$, then [Tot] proves that there is a natural isomorphism

$$\psi_X^n : H^n(X, \mathbb{Z}(n)) \xrightarrow{\sim} K_n^M(F)$$

between the degree-n, weight-n motivic cohomology and *Milnor K-theory*, which we recall is given in degree n by

$$K_n^M(F) := (F^\times \otimes \ldots \otimes F^\times)/I$$

where I is the degree-n part of the ideal in the free tensor algebra on $F^\times$ generated by $x \otimes (1 - x)$ for non-identity $x \in F^\times$. This map is given on the level of the Bloch cycle complex as follows: for the class of an irreducible closed subvariety $[Z] \in z^n(X, n)$, write $p : Z \to X$ for the projection map; then we set

$$\psi_X^n(z) := p_*([t_1 \otimes \ldots \otimes t_n]) \in K_n(F) \tag{2.1}$$

where p_* is the finite pushforward (also called *transfer map*) in Milnor K-theory (as defined in [BaTa, §5]) for the map p, where we recall the $t_\bullet$ are the coordinate functions on the simplicial cube.

When $n = 1$ above, the above discussion applies to more than fields: for any scheme X, the map $H^1(X, \mathbb{Z}(1)) \to \Gamma_X(\mathcal{O}_X^\times)$ given by sending $[Z] \mapsto p_*(t_1)$ (and zero if Z is not dominant over X) is an isomorphism identifying the degree-1 weight-1 motivic cohomology with the global units. This allows us to more generally consider cup products like

$$u_1 \smile \ldots \smile u_n \in H^n(X, \mathbb{Z}(n))$$

for units $u_1, \ldots, u_n$ on X. Following notational conventions in Milnor K-theory, we will also denote such cup products, as well as the associated tensors in Milnor K-theory, by the curly braces $\{u_1, \ldots, u_n\}$.

2.1.1. *Trace operators on cohomology of* $\mathbb{G}_m$. We observe the element $1 - z \in H^1(\mathbb{G}_m - \{1\}, \mathbb{Z}(1))$ satisfies

$$[a]_*(1-z) = 1-z$$

for any $a \in \mathbb{N}$, where $[a]_*$ denotes the restriction of the map $[a] : \mathbb{G}_m \to \mathbb{G}_m$ is the map induced by $z \mapsto z^a$, composed with pullback by the inclusion $\mathbb{G}_m - \{1\} \hookrightarrow \mathbb{G}_m$. In general, we use the same notation $[a] : \mathbb{G}_m^n \to \mathbb{G}_m^n$ for the same map coordinatewise $(z_1, \ldots, z_n) \mapsto (z_1^a, \ldots, z_n^a)$, and observe that also

$$[a]_*\{1-z_1, \ldots, 1-z_n\} = \{1-z_1, \ldots, 1-z_n\}.$$

For the submodule of the cohomology $\mathbb{G}_m^n$, or various subspaces thereof, we will write superscript (0) to mean the elements which are fixed in this way by all $[a]_*$; e.g.

$$\{1-z_1, \ldots, 1-z_n\} \in H^n((\mathbb{G}_m - \{1\})^n, \mathbb{Z}(n))^{(0)}.$$

Note that $H^n((\mathbb{G}_m - \{1\})^n, \mathbb{Z}(n))$ carries a pushforward action of $M_n(\mathbb{Z})$ commuting with all $[a]_*$; since the $[a]_*$ act invertibly, the $M_n(\mathbb{Z})$ actually induces a $\mathrm{GL}_n(\mathbb{Q})$-action on the trace-fixed submodule $H^n((\mathbb{G}_m - \{1\})^n, \mathbb{Z}(n))^{(0)}$. This applies to the trace-fixed cohomology of any subspace of $\mathbb{G}_m$ with a $M_n(\mathbb{Z})$-action.

2.2. **Symbols from spherical chain complexes.** We turn now to constructing a chain complex $\widetilde{\mathrm{Chains}}(n)$ with action by $\mathrm{GL}_n(\mathbb{Q})$, and show that it computes the reduced homology of the sphere S^{n-1} and is thus almost exact. In top degree, this complex will naturally parameterize special elements in the Milnor K-theory of the function field of $\mathbb{G}_m^n$, and we will abbreviate the top-degree module

$$\tilde{C}(n) := \widetilde{\mathrm{Chains}}(n)_n$$

The construction of this section directly generalizes [SV, §5], who consider the case $n = 2$, though we use slightly different conventions for the group action: we are taking *left* actions on spaces with a resulting pushforward left action on cohomology groups, while they consider right actions on spaces.

We identify $S^{n-1} = (\mathbb{R}^n - \{0\})/\mathbb{R}_{>0}^\times$ with the set of *rays* in an n-dimensional real vector space, and write $S^{n-1}(\mathbb{Q})$ for the subset of rational rays. Consider the poset $\mathcal{P}$ of linear triangulations T of S^{n-1} with vertices contained in $S^{n-1}(\mathbb{Q}) \subset S^{n-1}$, ordered by *refinement*: that is to say, we say $T_1 \geqslant T$ if every simplex of T is a union of simplices of T_1. Here, by *linear triangulation*, we mean every k-simplex is the intersection of a S^{n-1} with a dimension-$(k+1)$ simplicial polyhedral cone in $\mathbb{R}^n$ emanating from the origin. Notice that in particular, the great circle containing any given simplex can be recovered as the intersection of S^{n-1} with the span of the rays corresponding to the vertices of the simplex. As such, each of these corresponding subspaces are themselves rationally defined, since these rays are rational.

Given any geodesic k-simplex Δ with endpoints in $S^{n-1}(\mathbb{Q})$ with $k \leqslant n-1$, there exists a triangulation $T \in \mathcal{P}$ such that Δ is one of the triangles in T: indeed, for each face of Δ, take the corresponding great circle on S^{n-1}; these divide the sphere into polyhedral regions, which then can be barycentrically subdivided into simplices by picking a rational point in each interior. We have the following "upper bound" theorem on refinements of trinagulations.

Proposition 2.3. *Given any two triangulations $T_1, T_2 \in \mathcal{P}$, there exists a common upper bound for them, i.e., a triangulation $T \in \mathcal{P}$ refining both T_1 and T_2.*

Proof. Superimposing the triangulations T_1 and T_2, we obtain a decomposition of S^{n-1} into convex polyhedral cells. Each vertex of this superimposition is still in $S^{n-1}(\mathbb{Q})$, because any new vertices are the intersection points of rationally defined subspaces in $\mathbb{R}^n$ (corresponding to the great circles forming the intersecting faces) and hence themselves correspond to rational rays. Pick a rational point in interior of each polyhedral cell and subdivide it into simplices by joining the point to each $(n-2)$-face of the cell with a $(n-1)$-simplex; we obtain a new triangulation which still has all vertices in $S^{n-1}(\mathbb{Q})$. □

Write C_T for the augmented cellular chain complex of the simplicial complex T with

$$C_{T,i} = \mathbb{Z}\{T_i\}$$

for all $i \geqslant 0$, freely generated by the i-simplices endowed with their orientation coming from the globally oriented S^{n-1}. (We will consider the oppositely-oriented version $\overline{\Delta}$ of a simplex Δ to live in the chain complex, representing its additive inverse $[\overline{\Delta}] = -[\Delta]$.) The boundaries $C_{T,i+1} \xrightarrow{\partial} C_{T,i}$ given by the alternating face maps when $i \geqslant 0$. By "augmented", we mean that we set $C_{T,-1} = \mathbb{Z}$, with its incoming boundary map given by the degree map on 0-simplices. Then for any T,

$$H_{n-1}(C_{T,\bullet}) \cong \mathbb{Z}$$

is the only nontrivial homology group, since each simplicial complex is homotopy equivalent to S^{n-1}.

For a refinement T' of T, there is a pullback complex map

$$C_T \to C_{T'}$$

which is a quasi-isomorphism; these maps are functorial for compositions in the poset of triangulations. We can then define

$$\widetilde{\mathrm{Chains}}(n)_i := \varinjlim C_{T,i}[1] \tag{2.2}$$

as the colimit over all triangulations; by exactness of direct limits, $\widetilde{\mathrm{Chains}}(n)$ also only has homology $\mathbb{Z}$ in degree n; recall also that we abbreviate the top-degree term as $\tilde{C}(n)$.

Via the left action of $\mathrm{GL}_n(\mathbb{Q})$ on $S^{n-1}(\mathbb{Q})$ viewed as parameterizing rays in $\mathbb{R}^n$, the complex $\widetilde{\mathrm{Chains}}(n)$ takes a left action of $\mathrm{GL}_n(\mathbb{Q})$. In particular, if γ fixes a simplex Δ but $\det \gamma = -1$, then

$$\gamma \cdot [\Delta] = [\overline{\Delta}] = -[\Delta].$$

We now give generators and relations for $\widetilde{\mathrm{Chains}}(n)$. Given any tuple of independent rays $(m_1, m_2, \ldots, m_k)$ in $S^{n-1}(\mathbb{Q})$ lying in the same (open) hemisphere, which we will call an *acyclic* tuple, they span a unique oriented geodesic simplex (with orientation corresponding to the order), whose 1-frame is formed from the shorter segment of the great 1-circle through each pair of points. (By assumption, there are no antipodes, so there is no ambiguity.)

For any triangulation T containing this simplex, we write

$$[\Delta(m_1, m_2, \ldots, m_k)_T] \in C_{T,k-1}$$

for the class in homology; under any triangulation T' which refines T with the same property, the class $[\Delta(m_1, m_2, \ldots, m_k)_T]$ maps to $[\Delta(m_1, m_2, \ldots, m_k)_{T'}]$. Thus, we can speak unambiguously of a class in the direct limit

$$[\Delta(m_1, m_2, \ldots, m_k)] \in \widetilde{\mathrm{Chains}}(n)_k.$$

These classes generate $\widetilde{\mathrm{Chains}}(n)_i$ for each i, since any rational simplex can be subdivided into rational acyclic simplices.

We claim that the relations between the generators corresponding to these simplices are generated by subdivision of an acyclic simplex into subsimplices. Indeed, suppose we have an relation of k-simplex generators

$$[\Delta_1] + \ldots + [\Delta_t] = 0. \tag{2.3}$$

Superimposing all the great circles corresponding to the faces of the Δ_i and subdividing the resulting polyhedral cells as in the proof of 2.3, we can find a set of geodesic simplices $\Delta'_1, \ldots, \Delta'_s$, disjoint outside of their faces, such that each Δ_i is a union

$$\Delta_i = \bigsqcup_{j \in S_i} \Delta'_j.$$

Associated to this decomposition of simplices we have pushforward relation

$$[\Delta_i] = \sum_{j \in S_i} \Delta'_j. \tag{2.4}$$

Substituting these into (2.3), the resulting relation is between disjoint simplices, on which the coefficient of each simplex therefore must be zero. Hence relations of the form (2.4), coming from subdivision of an acyclic simplex, generate all relations as we claimed.

In fact, we can go even further, and generate all acyclic simplicial subdivision relations in terms of some easily classified ones:

Proposition 2.4. *Write $\mathbf{T}_k$ for the set of all acyclic positively oriented geodesic k-simplices with rational faces.*[3] *The module $\widetilde{\mathrm{Chains}}(n)_i$ for $i > 0$ is given by*

$$\mathbb{Z}\{\mathbf{T}_{i-1}\}/\mathcal{S}_{i-1}$$

where $\mathcal{S}_k$ is the module of acyclic k-simplicial subdivision relations, and is generated by stellar subdivisions *of rank r for $2 \leqslant r \leqslant k$ given as follows: fix an acyclic simplex $\Delta \in \Delta(m_1, \ldots, m_k)$ and a rational point m lying in the great circle corresponding to the face with vertices $(m_1, \ldots, m_r)$, but sharing some hemisphere of the great circle with all of them. Corresponding to this data we have the* stellar *subdivision relation*

$$[\Delta(m_1, \ldots, m_k)] = \sum_{i=1}^{r} [\Delta(m_1, \ldots, \hat{m_i}, m, m_{i+1}, \ldots, m_k)]. \tag{2.5}$$

Proof. From the discussion preceding the proposition, it remains only to prove that the stellar subdvision relations (2.5) generate all acyclic subdivision relations.

First, we say that a subdivision of a simplex Δ is a *sequentially stellar subdivision* if it can be obtained by a sequence of stellar subdivisions of Δ (and the resulting subsimplices at each sequential step). The relation corresponding to a sequentially stellar subdivision of an acyclic simplex is certainly generated by those of the implicated stellar subdivisions by repeated substitution.

We observe, therefore, that it suffices to show that any acyclic subdivision

$$\Delta = \Delta_1 \sqcup \ldots \sqcup \Delta_t$$

can be refined to a sequentially stellar subdivision of Δ, which is also a sequentially stellar subdivision of each Δ_i when restricted to it; this result in the field of combinatorial topology is due to [New]. □

We also write

$$\mathrm{Chains}(n) := \widetilde{\mathrm{Chains}}(n)/H_n(\widetilde{\mathrm{Chains}}(n))$$

[3] As before, we will also freely use negatively oriented simplices as generators, with the convention they are simply -1 times the positively oriented simplex.

for the exact quotient of $\widetilde{\mathrm{Chains}}(n)$ given by quotienting by top homology. We use the analogous abbreviation

$$C(n) := \mathrm{Chains}(n)_n$$

for the top-degree module.

2.3. **Combinatorial cohomology classes for** GL_n. We now construct "abstract" cohomology classes for $\mathrm{GL}_n(\mathbb{Q})$ valued in symbols in the symbol complex, whose realization in Milnor K-theory (and later, differential forms) will yield our arithmetic classes of interest. We use the following standard construction in group cohomology, whose proof we sketch:

Lemma 2.5. *If a group G acts on an exact complex $C_\bullet$ supported in degrees $[0, n]$, then we have a natural map on cohomology*

$$C_0^G \to H^{n-1}(G, C_n)$$

inhomogeneous cocycle representatives of which can be constructed as follows: associated to $e \in C_0^G$, pick a lift ℓ_1 of e to C_1, and consider the 1-cochain

$$\gamma \mapsto (\gamma - 1)\ell_1 \in C^1(G, C_1).$$

By exactness, this is the boundary of an element $\ell_2 \in C^1(G, C_2)$; we take the chain coboundary $\partial \ell_2 \in C^2(G, C_2)$ which again lifts to $\ell_3 \in C^2(G, C_3)$, etc. The lift $\ell_n \in C^{n-1}(G, C_n)$ is a cocycle representing the image of e in $H^{n-1}(G, C_n)$.

Proof. The map $C_0^G \to H^{n-1}(G, C_n)$ is the composition of connecting homomorphisms

$$C_0 \to H^1(G, \ker \partial_1) = H^1(G, \mathrm{im}\partial_2) \tag{2.6}$$

$$\to H^2(G, \ker \partial_2) = H^2(G, \mathrm{im}\partial_3) \tag{2.7}$$

$$\to \dots \tag{2.8}$$

$$\to H^{n-1}(G, \ker \partial_n) = H^{n-1}(G, C_n). \tag{2.9}$$

for the G-short exact sequences

$$\ker \partial_i \xrightarrow{\partial_{i+1}} C_i \xrightarrow{\partial_i} \mathrm{im}\partial_i.$$

Unwinding the definition of each of these maps yields precisely the process of iterated lifting described in the lemma. □

We apply the lemma to the action of $\mathrm{GL}_n(\mathbb{Q})$ on $\mathrm{Chains}(n)$, using the element $e \in \mathbb{Z}$ in degree zero of each respective complex. This affords us a cohomology class

$$\Theta^{S^{n-1}}(n) \in H^{n-1}(\mathrm{GL}_n(\mathbb{Q}), C(n)). \tag{2.10}$$

2.3.1. *Explicit cocycle representatives.* We can apply Lemma 2.5 to obtain cocycle representatives for $\Theta^{S^{n-1}}(n)$.

To write down our explicit cocycle, it will be useful to specify an extension of the previous notation $[\Delta(m_1, \dots, m_k)]$ to *any* tuple of rays $(m_1, \dots, m_k)$, even tuples failing to be acyclic or independent. Thus, we define a Δ-*extension* E to be a collection of classes[4]

$$[\Delta_E(m_1, \dots, m_k)] \in \mathrm{Chains}(n)_k$$

for arbitrary tuples of rays $(m_1, \dots, m_k)$ satisfying the following properties:

[4] This is somewhat abusive notation, since we are not necessarily saying that $[\Delta_E(m_1, \dots, m_k)]$ is actually the class of a particular geodesic simplex when the tuple is not linearly independent.

(1) If the tuple $(m_1, \ldots, m_k)$ is independent, then

$$[\Delta_E(m_1, \ldots, m_k)] = [\Delta(m_1, \ldots, m_k)]$$

as we have previously defined it.

(2) $\gamma[\Delta_E(m_1, \ldots, m_k)] = [\Delta_E(\gamma m_1, \ldots, \gamma m_k)]$ for all $\gamma \in \mathrm{GL}_n(\mathbb{Q})$.

(3) The image of $[\Delta_E(m_1, \ldots, m_k)]$ under the face map is

$$\sum_i (-1)^{i-1}[\Delta_E(m_1, \ldots, \hat{m}_i, \ldots m_k)].$$

(4) If $(m_1, \ldots m_k)$ is a dependent acyclic tuple, then $[\Delta_E(m_1, \ldots, m_k)]$ is zero.

We will call the data of such an extension a Δ-*extension*; all Δ-extensions by definition agree on linearly independent tuples.

Proposition 2.6. *Δ-extensions exist.*

Proof. We show how to construct such an E inductively on the corank of $(m_1, \ldots, m_k)$: that is, on the difference $k - r$ between the number of rays and the rank of their span. When $k = r$, the definition of $[\Delta_E(m_1, \ldots, m_k)]$ is forced on us by (1); these definitions certainly satisfy (2) and (3), and (4) is not applicable. This furnishes the base case.

We will now first complete the inductive step so as to fulfill (2), (3), then return to analyze how one fulfills (4) in different inductive steps, as fulfilling (4) cannot be analyzed uniformly across all steps.

Indeed, to satisfy (2), all we need to do is construct $\Delta_E(m_1, \ldots, m_k)$ for an arbitrary representative of each $\mathrm{GL}_n(\mathbb{Q})$-orbit of tuples and extend by group translation, since the conditions (3) and (4) are certainly translation-invariant. Thus, assume that we have constructed Δ_E for corank up to $i - 1$, and we wish to construct it for tuples of corank i. Then the point is that

$$\sum_i (-1)^{i-1}[\Delta_E(m_1, \ldots, \hat{m}_i, \ldots m_k)] \tag{2.11}$$

has boundary zero, so by exactness of $\mathrm{Chains}(n)$ it is possible to pick some chain lifting it under the boundary map.

We claim that $(m_1, \ldots, m_k)$ is acyclic, we can pick the extension so that (2.11) is identically zero. Indeed, in the first inductive step $i = 1$, (2.11) is precisely a stellar subdivision relation, so it vanishes and we can pick the lift to be zero. When $i > 1$, then inductively all the terms of (2.11) are acyclic dependent tuples, which by the inductive hypothesis are zero, and hence we can pick zero as a lift under the boundary map. □

To any Δ-extension, there corresponds an explicit cocycle:

Theorem 2.7. *For any $(n-1)$-tuple of matrices $\underline{\gamma} = (\gamma_1, \ldots, \gamma_{n-1})$, write $c_i(\underline{\gamma})$ for*

$$\gamma_1\gamma_2 \cdots \gamma_i e_1$$

for $i \geqslant 0$, where $e_1 = (1, 0, \ldots, 0)$; this is equivalently the first column of the product matrix written above. Now fix any Δ-extension E; for such an extension, we define a $(n-1)$-cochain $\theta_E^{S^{n-1}}(n)$ by

$$\underline{\gamma} \to [\Delta_E[\underline{\gamma}]] := [\Delta_E(c_{n-1}(\underline{\gamma}), c_{n-2}(\underline{\gamma}), \ldots, c_0(\underline{\gamma}))]. \tag{2.12}$$

(In particular, this is the zero class if the simplex in question is degenerate.) Then $\theta_E^{S^{n-1}}(n)$ is a cocycle representative for $\Theta^{S^{n-1}}(n)$.

Proof. This follows immediately from the properties of a Δ-extension and the lifting process of 2.5: we lift $1 \in \mathbb{Z}$ to e_1, whose group coboundary is the 1-cochain $\gamma \mapsto (\gamma - 1)e_1$ which lifts to

$$\gamma \mapsto [\Delta_E(c_1(\gamma), c_0(\gamma))].$$

The group coboundary of this lifts to the 2-cochain

$$\gamma \mapsto [\Delta_E(c_2(\gamma_1, \gamma_2), c_1(\gamma_1, \gamma_2), c_0(\gamma_1, \gamma_2))]$$

and so on. □

Remark 2.8. The construction of Shintani cocycles (though not the motivic setting) by [Hill] (and used by [LP]) introduces a lexicographical order on lines in order to make the cocycle property work. From our perspective, the data of this choice of order is the same data one needs to specify a certain Δ-extension, though one must be careful: their construction is actually "dual" to ours in the sense specified later in Section 4.2, so this is not quite precise. To be more exact, making free use of the language and concepts of that (later) section, their lexicographic order is used to specify choices of lower-dimensional conical faces to include, which *does* correspond under conical duality corresponds to the degenerate wedge classes chosen in a Δ-extension.

From the point of view of the cohomology class, the particular values of the cocycle (on non-acyclic tuples of group elements only) resulting from these auxiliary choices are thus not of independent significance. We thank Jeehoon Park for pointing this out.

Remark 2.9. Observe that the ambiguities in the choices of Δ-extensions only matter for tuples such that the lines associated to $c_0, \ldots, c_{n-1}$ are dependent. In particular, if $\gamma_1, \ldots, \gamma_{n-1}$ are generators of an anisotropic torus of rank $n-1$ inside $\mathrm{GL}_n(\mathbb{Z})$, then the value of our cocycle on $(\gamma_1, \ldots, \gamma_{n-1})$ is independent of the Δ-extension.

2.3.2. *Lifting the symbol-valued cocycles.* The cohomology class $\Theta^{S^{n-1}}(n)$ is valued in $C(n)$; in particular, we have quotiented by the fundamental class of the sphere. As we will see in section 3.1, the realizations of these modules will live in a *quotient* of motivic cohomology groups of $\mathbb{G}_m^n$ by a rank-one submodule coming from the "orientation obstruction" of the fundamental class. In [SV, §5], the authors show that their cocycle (which agrees with ours for the case $\mathrm{SL}_2(\mathbb{Z})$) can be lifted over this obstruction after inverting 6; in this section, we indicate how this generalizes. In particular, we will to lift our cocycles to be valued in $\tilde{C}(n)$, i.e lift over the copy of the rank-1 free module $\mathbb{Z}$ corresponding to the fundamental class of S^{n-1}.

To begin, note that applying the lifting process of Lemma 2.5 to the length-$(n+2)$ exact complex of $\mathrm{GL}_n(\mathbb{Q})$-modules

$$\mathbb{Z}(\mathrm{sgn}) \to \widetilde{\mathrm{Chains}}(n)_\bullet \tag{2.13}$$

in the same way as we did with $\mathrm{Chains}(n)$ yields a cocycle $\varepsilon_n^E \in C^n(\mathrm{GL}_n(\mathbb{Q}), \mathbb{Z}(\mathrm{sgn}))$.

Proposition 2.10. *The cocycle ε_n^E represents the Euler class for the standard representation of* $\mathrm{GL}_n(\mathbb{Q})$.

Proof. The complex $\widetilde{\mathrm{Chains}}(n)$ is the reduced homology complex of an ind-simplicial model of the $(n-1)$-sphere; since this is a closed manifold, we can view it also by Poincaré duality, as computing cohomology in the complementary degree. Then the double complex $C^\bullet(\mathrm{GL}_n(\mathbb{Q}), \widetilde{\mathrm{Chains}}(n)_\bullet)$ computes the equivariant cohomology of S^{n-1}. By Lemma 2.5, the sum of the lifts associated to E

$$\ell_1 + \ldots + \ell_{n+1}$$

(using the notation of the lemma) for the lifts of $1 \in \mathbb{Z}$ in the augmentation for this complex then is a representative for the Thom class of the associated sphere bundle, since the Thom class is dual to the zero section. Pulling back to the base (i.e. a point with the trivial $\mathrm{GL}_n(\mathbb{Q})$-action; equivalently, $B\mathrm{GL}_n(\mathbb{Q})$) by the zero section kills all terms except $\varepsilon_n^E = \ell_{n+1}$, which is therefore a representative of the Euler class. $\square$

Suppose now that Γ is an S-arithmetic subgroup of $\mathrm{GL}_n(\mathbb{Q})$, where S is any subset of primes which has nonempty complement. The following vanishing is a theorem of Sullivan [Sul]:

Theorem 2.11. *The cocycle ε_n, restricted to Γ, is a coboundary after multiplying by the greatest common denominator $d_{n,S}$ of $m^n(m^n-1)$, as m ranges over all integers divisible only by primes not in S. For S empty, $d_n := d_{n,\varnothing}$ is twice the denominator of the nth Bernoulli number.*

The short exact sequence

$$0 \to \mathbb{Z}(\mathrm{sgn}) \to \tilde{C}(n) \to C(n) \to 0$$

yields a long exact sequence in cohomology

$$\ldots \to H^{n-1}(\Gamma, \mathbb{Z}(\mathrm{sgn})) \to H^{n-1}(\Gamma, \tilde{C}(n)) \to H^{n-1}(\Gamma, C(n)) \to H^n(\Gamma, \mathbb{Z}(\mathrm{sgn})) \to \ldots$$

After inverting $d_{n,S}$, the image of $\Theta^{S^{n-1}}(n)$ (i.e. the Euler class) in the rightmost term vanishes, and hence it lifts non-uniquely to be valued in $\tilde{C}(n)$, and the set of lifts of is a torsor under the image of $d_{n,S}^{-1} \cdot H^{n-1}(\Gamma, \mathbb{Z}(\mathrm{sgn}))$. We thus find that:

Corollary 2.12. *After inverting $d_{n,S}$ and restricting to Γ, representatives for these lifts are given by*

$$\underline{\gamma} \mapsto [\Delta_E(\underline{\gamma})] - \phi(\underline{\gamma})$$

where $\phi(\underline{\gamma})$ ranges over primitives of ε_n^E.

Write $\theta_{E,\phi}^{S^{n-1}}(n)$ for the cocycle corresponding to the $(n-1)$-cochain ϕ transgressing the Euler cocycle; the corresponding class is

$$\Theta_\phi^{S^{n-1}}(n) = [\theta_{E,\phi}^{S^{n-1}}(n)] \in d_{n,S}^{-1} \cdot H^{n-1}(\Gamma, \tilde{C}(n)). \tag{2.14}$$

Notice that the difference of two such ϕ is a cocycle, so the space of possible lifts over the Euler class is a torsor under $d_{n,S}^{-1} \cdot H^{n-1}(\mathrm{GL}_n(\mathbb{Q}), \mathbb{Z}(\mathrm{sgn}))$.

Remark 2.13. As mentioned earlier, this generalizes the construction in [SV, §5] of a canonical lift of the analogue to the cocycle $\theta^{S^{n-1}}(2)$ for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. (When $n = 2$, the choice of E is immaterial, so we omit it from the notation.) In this case, $H^1(\Gamma) = H^2(\Gamma) = 0$ after inverting 6, so there is no ambiguity of lift, and the authors consequently find an explicit distinguished primitive of the Euler class ε_2 to give them a canonical lift. This raises the question whether there is some more canonical choice of lift in general even when $H^{n-1}(\Gamma)$ does not vanish; we do not know the answer in general. However, when n is odd, one can obtain a canonical lift up to 2-torsion by applying the projector $[-1]_* + 1$, which annihilates the Euler class but acts as the scalar 2 on our cocycles.

2.3.3. *The Steinberg quotient.* We end this section by explaining how $\theta_E^{S^{n-1}}(n)$ can be modified into a parabolic cocycle if one quotients out by extra relations; this will also remove the need for a choice of Δ-extension E in the lifting process. After defining our realizations of these symbols, this will correspond to quotienting by certain elements in K-theory; see subsection 3.1.1 for details. This quotient will also be relevant for the application to Sharifi's conjectures.

Let $\mathrm{St}(n)$ be the *Orlik-Solomon complex* [OS] defined as follows: it is the graded-commutative algebra generated in degree 1 by symbols $[\ell]$ for $\ell \in \mathbb{P}^{n-1}(\mathbb{Q})$, with relations generated multiplicatively by the dependence relations

$$\partial([\ell_1] \wedge \ldots \wedge [\ell_k]) = 0$$

for any lines $\ell_1, \ldots, \ell_k$ spanning a space of rank strictly less than k. Here, the differential-graded structure is defined via $\partial[\ell] = 1$ and extended by the graded Leibniz rule. The relations generated as above are closed under the differential, and the resulting differential-graded algebra is exact. (More generally, for any configuration of lines, not just the set of *all* rational ones, one can define an Orlik-Solomon complex with the same properties; this will be used later in Section 4.3.)

Then we have a $\mathrm{GL}_n(\mathbb{Q})$-equivariant map of complexes

$$\mathrm{Chains}(n)_i \twoheadrightarrow \mathrm{OS}(n)_i \tag{2.15}$$

sending

$$\Delta(r_1, \ldots, r_k) \mapsto [\mathbb{Q}r_1] \wedge \ldots \wedge [\mathbb{Q}r_k].$$

This map (2.15) is well-defined, since the Orlik-Solomon complex obeys alternation in the vertices, and stellar subdivision relations simply become dependence relations in the image.

In top degree, the map (2.15) corresponds to the $\mathrm{GL}_n(\mathbb{Q})$-equivariant quotient of the top spherical chains

$$R_{St} : C(n) \twoheadrightarrow \mathrm{St}(n) := \mathrm{OS}(n)_n \tag{2.16}$$

where $\mathrm{St}(n)$ is our notation for a $\mathrm{GL}_n(\mathbb{Q})$-module often called the *Steinberg representation*. From the definition of the Orlik-Solomon algebra, we see that it can be described as generated by symbols $[\ell_1] \wedge \ldots \wedge [\ell_n]$ where $\ell_1, \ldots, \ell_n \in \mathbb{P}^{n-1}(\mathbb{Q})$, quotiented by the relations

(1) $[\ell_1] \wedge \ldots \wedge [\ell_n] = 0$ if the lines do not span $\mathbb{Q}^n$, and
(2) For any $\ell_0, \ldots, \ell_n$, the dependence relation

$$\sum_{i=0}^{n} (-1)^i [\ell_0] \wedge \ldots \wedge [\hat{\ell}_i] \wedge \ldots \wedge [\ell_n] = 0.$$

From this description, one sees that the pushforward of $\theta_E^{S^{n-1}}(n)$ along (2.16) is therefore independent of the choice of E, since all dependent tuples simply are sent to zero (as all independent tuples bound some acyclic simplex). Hence we get a cocycle

$$\theta^{St}(n) := (R_{St})_* \theta_E^{S^{n-1}}(n), (\gamma_1, \ldots, \gamma_{n-1}) \mapsto [c_{n-1}(\underline{\gamma})] \wedge \ldots \wedge [c_0(\underline{\gamma})]$$

independent of E, representing a class

$$\Theta^{St}(n) \in H^{n-1}(\mathrm{GL}_n(\mathbb{Q}), \mathrm{St}(n))$$

which one can see from relation (1) is parabolic.[5]

We conclude this section with the following description of the kernel of (2.16), which will be useful when considering realizations:

Lemma 2.14. *The kernel of the map* (2.16) *is generated by "wedge" classes of the form*

$$\Delta(r_1, \ldots, r_n) - \Delta(-r_1, r_2, \ldots, r_n)$$

[5]After restricting to $\mathrm{SL}_n(\mathbb{Z})$, this cocycle is in fact the universal parabolic cocycle coming from Bieri-Eckman duality, as the Steinberg module is the dualizing module for $\mathrm{SL}_n(\mathbb{Z})$.

for independent tuples $(r_1, \dots, r_n)$.

Proof. The top-dimensional spherical chains are generated by acyclic simplices together with stellar subdivision relations, while the Steinberg module is generated by independent tuples modulo dependence relations. If one imposes all the identification of spherical simplices

$$\Delta(r_1, \dots, r_n) \sim \Delta(\pm r_1, \dots, \pm r_n),$$

for any combination of signs, then the relations resulting from the stellar subdivision relations are precisely the dependence relations. These identifications can be deduced, by (anti)symmetry, from the identifications

$$\Delta(r_1, \dots, r_n) \sim \Delta(-r_1, r_2, \dots, r_n)$$

since we can then change one sign at a time and bootstrap to the general case. □

3. Construction and application of the motivic cocycles

To

3.1. **Realization map for top-dimensional chains.** We now turn to constructing the realization map associating classes in motivic cohomology/Milnor K-theory to symbols. This section will be devoted to proving the following theorem:

Theorem 3.1. *There exists a map of* $\mathrm{GL}_n(\mathbb{Q})$*-modules*

$$\tilde{\rho} : \widetilde{\mathrm{Chains}}(n)_n \to K_M^n(k(\mathbb{G}_m^n))^{(0)}.$$

defined on generators by[6]

$$[\Delta(m_1, \dots, m_n)] \mapsto \begin{pmatrix} m_1 & \dots & m_n \end{pmatrix}_* \{1 - z_1, \dots, 1 - z_n\} \in K_n^M(k(\mathbb{G}_m^n))^{(0)} \tag{3.1}$$

The image of the fundamental class in $\mathbb{Z} \cong H_n(\widetilde{\mathrm{Chains}}(n)) \subset \widetilde{\mathrm{Chains}}(n)$ *is the class* $\{-z_1, \dots, -z_n\}$*, so* $\tilde{\rho}$ *descends to a map*

$$\rho : \mathrm{Chains}(n)_n \to K_M^n(k(\mathbb{G}_m^n))^{(0)}/\{-z_1, \dots, -z_n\}.$$

The hard part of the theorem is proving the relations, so we first view f_k as a map from $\mathbb{Z}\{\mathbf{T}_{n+1}\}$.

We note that replacing m_i by a scalar multiple is immaterial because each $1 - z_i$ is invariant under $[a]_* : \mathbb{G}_m \to \mathbb{G}_m$ in the corresponding coordinate, so pre-composing $\begin{pmatrix} m_1 & \dots & m_i \end{pmatrix}_*$ with these isogenies does not change the definition of ρ. The $\mathrm{GL}_n(\mathbb{Q})$-equivariance of the definition is then completely formal, from functoriality of pushforwards.

It remains to check that the relations in $\mathcal{S}_\bullet$ between the classes of simplices hold. By Proposition 2.4, it suffices to check the acyclic stellar subdvision relations.

Proposition 3.2. *For any integer* $2 \leqslant r \leqslant n$*, each acyclic independent tuple* $\underline{m} = (m_1, \dots, m_n)$ *of rays and ray* m *lying on the great circle corresponding to the face spanned by* $(m_1, \dots, m_n)$ *and sharing some hemisphere with all of them, the relation*

$$\rho[\Delta(m_1, \dots, m_k)] = \sum_{i=1}^{r} \rho[\Delta(m_1, \dots, \hat{m_i}, m, m_{i+1}, \dots, m_n)] \tag{3.2}$$

coming from (2.5) *holds.*

[6]Here, the matrix columns can be any non-zero vector in the ray; see discussion following the theorem statemnt for why this is well-defined.

Proof. We may reduce to the case $r = n$ as follows: the claimed relation can be written as

$$\begin{pmatrix} m_1 & \dots & m_n \end{pmatrix}_* \{1-z_1, \dots, 1-z_n\} = \sum_{i=1}^{r} \begin{pmatrix} m_1 & \dots & \hat{m}_i & m & \dots & m_k \end{pmatrix}_* \{1-z_1, \dots, 1-z_n\}.$$

The left-hand side factors as the cup product

$$\begin{pmatrix} m_1 & \dots & m_r \end{pmatrix}_* \{1-z_1, \dots, 1-z_r\} \smile \begin{pmatrix} m_{r+1} & \dots & m_n \end{pmatrix}_* \{1-z_{r+1}, \dots, 1-z_n\}$$

and the right-hand side as

$$\left(\sum_{i=1}^{r} \begin{pmatrix} m_1 & \dots & \hat{m}_i & m & \dots & m_r \end{pmatrix}_* \{1-z_1, \dots, 1-z_r\}\right) \smile \begin{pmatrix} m_{r+1} & \dots & m_n \end{pmatrix}_* \{1-z_{r+1}, \dots, 1-z_n\}$$

so it suffices to prove that

$$\begin{pmatrix} m_1 & \dots & m_r \end{pmatrix}_* \{1-z_1, \dots, 1-z_r\} = \sum_{i=1}^{r} \begin{pmatrix} m_1 & \dots & \hat{m}_i & m & \dots & m_r \end{pmatrix}_* \{1-z_1, \dots, 1-z_r\}$$

But this is the pullback of a top-rank stellar relation from the quotient $\mathbb{G}_m^n \twoheadrightarrow \mathbb{G}_m^n/G$, for G the image of

$$\begin{pmatrix} m_{r+1} & \dots & m_n \end{pmatrix} : \mathbb{G}_m^r \to \mathbb{G}_m^n.$$

We therefore henceforth assume that $r = n$, and need to prove the relation

$$\begin{pmatrix} m_1 & \dots & m_n \end{pmatrix}_* \{1-z_1, \dots, 1-z_n\} - \sum_{i=1}^{n} \begin{pmatrix} m_1 & \dots & \hat{m}_i & m & \dots & m_n \end{pmatrix}_* \{1-z_1, \dots, 1-z_n\} = 0.$$

Taking the Bloch cycle description

$$K_n^M(k(\mathbb{G}_m^n)) \hookrightarrow z^n(k(\mathbb{G}_m^n) \times \square^n)/\partial z^{n+1}(k(\mathbb{G}_m^n) \times \square^{n+1})$$

we see that it suffices to show that

$$\sum_{i=1}^{n+1} (-1)^i \begin{pmatrix} m_1 & \dots & \hat{m}_i & \dots & m_{n+1} \end{pmatrix}_* \Gamma(1-z_1, \dots, 1-z_n) \in \partial z^{n+1}(k(\mathbb{G}_m^n) \times \square^{n+1}). \tag{3.3}$$

for all acyclic tuples of rays $(m_1, \dots, m_{n+1})$ such that each sub-n-tuple is full rank. Formally, (3.3) is the image under the cubical face maps of

$$\begin{pmatrix} m_1 & \dots & m_{n+1} \end{pmatrix}_* \Gamma(1-z_1, \dots, 1-z_{n+1}) \in \partial z^{n+1}(k(\mathbb{G}_m^n) \times \square^{n+1}) \tag{3.4}$$

where the matrix denotes the map

$$\begin{pmatrix} m_1 & \dots & m_{n+1} \end{pmatrix} : \mathbb{G}_m^{n+1} \to \mathbb{G}_m^n. \tag{3.5}$$

However, this seems not to use the acyclicity condition; what gives? The problem is that (3.5) is not a finite map, so the pushed forward cycle may have improper intersection with the faces. Indeed, we claim that (3.4) intersects all faces properly exactly when $(m_1, \dots, m_{n+1})$ is acyclic. We will prove this on the level of cycles in

$$\mathbb{G}_m^n \times \square^{n+1}$$

since this certainly implies it for the restriction to the generic point. Write M for the matrix $(m_1, \dots, m_{n+1})$, and define the $(n+1)$-variable monomials

$$p_j(z_1, \dots, z_{n+1}) = \prod_{i=1}^{n+1} z_i^{M_{j,i}}$$

whose exponents are the jth row of M. Then in $\mathbb{G}_m^n \times \square^{n+1}$, the cycle (3.4) can be described as the closure of the locus

$$(p_1, p_2, \dots, p_n, 1 - z_1, \dots, 1 - z_{n+1}).$$

Its intersection with a codimension-d cubical face is then indexed by a labelled subset $I \in \{1, \dots, n+1\}$ of cardinality d, with each element i of the subset labelled by $\ell_I(i) \in \{0, \infty\}$; this face is given by the intersection of the cycle with the locus

$$\bigcap_{i \in I} \{1 - z_i = \ell_I(i)\}$$

i.e. fixing each z_i corresponding to the indexing set to be either 1 or ∞. We wish to check, for each I, whether or not this intersection has the correct codimension, i.e. codimension $n + |I|$; the cycle (3.4) meets all faces properly if and only if all codimensions are correct.

First consider the case where I has at least one label of 0; without loss of generality, we assume that it corresponds to $1 \in I$, i.e. the relation $1 - z_1 = 1 \Leftrightarrow z_1 = 0$. The intersection of (3.4) with $\{z_1 = 0\}$ is then the closure of the locus

$$(p_1(z_1 = 0), \dots, p_n(z_1 = 0), 0, 1 - z_2, \dots, 1 - z_n). \tag{3.6}$$

where the notation indicates that we plug in 0 in the ith place. Write M' for the submatrix of M given by deleting the first column; by assumption, it has full rank n. Viewing M' as associated to a map

$$\mathbb{G}_m^n \times \square^n \to \mathbb{G}_m^n \times \square^{n+1}$$

by its natural action on the toric part, and by inclusion $\square^n \hookrightarrow \square^{n+1}$ in the last n coordinates, we see that the locus (3.6) is then the finite pushforward

$$M'_* \Gamma(1 - z_1, \dots, 1 - z_n). \tag{3.7}$$

The intersection of (3.4) with a face corresponding to I is then the intersection of (3.7) with the face corresponding to the labelled set $I \backslash \{1\}$. But (3.7) is a finite pushforward of a cycle meeting all faces properly, so we conclude that (3.4) meets the face corresponding to I properly as well. Thus, any face with at least one label of 0 always intersects (3.4) properly; it therefore suffices to check the intersection with faces labelled only with ∞s.

We claim that if $(m_1, \dots, m_{n+1})$ is acyclic, this intersection is always empty, and thus trivially proper.[7] Note that the property of having empty intersection with the ∞-labelled faces is invariant under left multiplication of M by elements of $\mathrm{GL}_n(\mathbb{Q}) \cap M_n(\mathbb{Z})$. Via left multiplication by such a matrix, we can always turn the first n columns of the matrix into scalar multiples of the standard basis. Thus, it suffices to check matrices of the form

$$\begin{pmatrix} x_1 & & & & y_1 \\ & x_2 & & & y_2 \\ & & \dots & & \dots \\ & & & x_n & y_n \end{pmatrix}$$

where each of the x_i and y_i must be nonzero by the assumption that every n-by-n submatrix of M is full rank, and $\mathrm{sgn}(x_i) = \mathrm{sgn}(y_i)$ for at least one i by the acyclicity assumption.

[7] When $(m_1, \dots, m_{n+1})$ fails to be acyclic, the corresponding stellar simplicial relation should not hold, and thus (3.4) must not meet all faces properly. The simplest example of what happens in this case: if $n = 1$, $m_1 = 1$, and $m_2 = -1$, then the locus (3.4) is the closure of $(z_1 z_2^{-1}, 1 - z_1, 1 - z_2)$. Its intersection with the unique codimension-two face labelled with two ∞s should therefore be codimension 3, i.e. zero-dimensional. However, the points of the curve parameterized by (g, ∞, ∞) are in the closure for all $g \in \mathbb{G}_m$, since the point corresponding to each fixed g is in the closure (as $t \to \infty$) of the curve $z_1 = gt$, $z_2 = t$ with free parameter t.

In this case, suppose without loss of generality that $I = \{1, 2, \ldots, k\}$ for some $k \leqslant n$, with $\ell(j) = \infty$ for each $j \in I$. Assume now for the sake of contradiction that (3.4) intersects the face corresponding to I nontrivially; in particular, suppose it contains a point with $z_1 = z_2 = \ldots = z_k = \infty$ but

$$p_i = z_i^{x_i} z_{n+1}^{y_i} \in \mathbb{G}_m(\overline{k})$$

for $\operatorname{sgn}(x_i) = \operatorname{sgn}(y_i)$. If $i \in \{1, 2, \ldots, k\}$, this is impossible, since at this point we must have $z_{n+1} = 0 \Rightarrow 1 - z_{n+1} = 1$, which means our point fails to lie in the algebraic cube $\square^{n+1}$. Otherwise, x_j and y_j have opposite signs for $j = 1, 2, \ldots, k$, meaning that $z_{n+1} = \infty$ as well at this point. But then

$$p_i = z_i^{x_i} z_{n+1}^{y_i} \in \mathbb{G}_m(\overline{k})$$

implies that $z_i = 0 \Rightarrow 1 - z_i = 1$, again a contradiction. We conclude the intersection with the face corresponding to I is in fact empty, as desired.

It remains only to show that the fundamental class of S^{n-1} is sent to the generator

$$\{-z_1, \ldots, -z_n\}.$$

Indeed, we can decompose the fundamental class as a sum of simplicial orthants

$$[S^{n-1}] = \sum_{I \in \{\pm 1\}^n} [\Delta(I)]$$

where $\Delta(I)$ is the simplex corresponding to $\sigma(I)(I_1 \cdot e_1, \ldots, I_n \cdot e_n)$ where $\sigma(I)$ is an arbitrary even permutation if $(I_1 \cdot e_1, \ldots, I_n \cdot e_n)$ is positively oriented, is an arbitrary odd permutation otherwise. Under f_n, this is sent to the sum

$$\sum_{I \in \{\pm 1\}^n} \sigma\{1 - z_1^{I_1}, \ldots, 1 - z_n^{I_n}\} = \left\{\frac{1 - z_1}{1 - z_1^{-1}}, \ldots \frac{1 - z_n}{1 - z_n^{-1}}\right\} = \{-z_1, \ldots, -z_n\}.$$

□

With the realization map in hand, the proof of Theorem 1.1 is complete. In particular, we obtain the following classes:

$$\Theta(n) = \rho_* \Theta^{S^{n-1}}(n) \in H^{n-1}(\mathrm{GL}_n(\mathbb{Q}), (K_n^M(k(\mathbb{G}_m^n))/\{-z_1, \ldots, -z_n\})^{(0)}).$$

and, if $\Gamma \subset \mathrm{GL}_n(\mathbb{Q})$ is S-arithmetic for a co-nonempty set of primes S, for any transgression ϕ of ε_n a class

$$\Theta_\phi(n) = \rho_* \Theta_\phi^{S^{n-1}}(n) \in H^{n-1}(\Gamma, K_n^M(k(\mathbb{G}_m^n))^{(0)})[d_{n,S}^{-1}].$$

For any Δ-extension E, the cocycle representative $\theta_E(n) := \rho_* \theta_E^{S^{n-1}}(n)$ for the former class is given by

$$(\gamma_1, \ldots, \gamma_{n-1}) \mapsto \begin{pmatrix} c_1 & \ldots & c_k \end{pmatrix}_* \{1 - z_1, 1 - z_2, \ldots, 1 - z_n\} \in K_n^M(k(\mathbb{G}_m))/\{-z_1, \ldots, -z_n\}$$

with $c_i = \gamma_i \ldots \gamma_1 e_1$ whenever these columns are independent; for any fixed Δ-extension E, one can equally in principle work out the image of any tuple with non-independent such c_i, though we do not currently see a systematic way to do this. Similarly, the latter class is represented by $\theta_{E,\phi}(n) := \rho_* \theta_{E,\phi}^{S^{n-1}}(n)$ and under the same assumptions sends

$$(\gamma_1, \ldots, \gamma_{n-1}) \mapsto \begin{pmatrix} c_1 & \ldots & c_k \end{pmatrix}_* \{1 - z_1, 1 - z_2, \ldots, 1 - z_n\} - \phi(\underline{\gamma})\{-z_1, \ldots, -z_n\} \tag{3.8}$$

in $K_n^M(k(\mathbb{G}_m))[d_{n,S}^{-1}]$.

3.1.1. *Modular symbols from the Steinberg quotient.* We now describe the realization of the parabolic, Steinberg module-valued, cocycle of section 2.3.3. This amounts to determining the image of the kernel of (2.16) under the map f, which by Lemma 2.14, is generated by the $\mathrm{GL}_2(\mathbb{Q})$-orbit of

$$\{1-z_1, 1-z_2, \ldots, 1-z_n\} - \{1-z_1^{-1}, 1-z_2, \ldots, 1-z_n\} = \{-z_1, 1-z_2, \ldots, 1-z_n\}.$$

We thus see it suffices to quotient out by the degree-n part of the ideal I of the Milnor K-theory ring generated by the symbols $-z_i \in K_1^M(k(\mathbb{G}_m))$, whereupon we obtain a $\mathrm{GL}_2(\mathbb{Q})$-equivariant map

$$\Theta^{MS}(n) : \mathrm{St}(n) \to (K_n^M(k(\mathbb{G}_m))/I)^{(0)}$$

sending

$$[\ell_1] \wedge \ldots \wedge [\ell_n] \mapsto \begin{pmatrix} \ell_1 & \ldots & \ell_k \end{pmatrix}_* [\{1-z_1, 1-z_2, \ldots, 1-z_n\}] \in K_n^M(k(\mathbb{G}_m))/I.$$

From this modular symbol, one can also deduce an explicit parabolic cocycle

$$H^{n-1}(\mathrm{GL}_2(\mathbb{Q}), (K_n^M(k(\mathbb{G}_m))/I)^{(0)}$$

represented by

$$(\gamma_1, \ldots, \gamma_{n-1}) \mapsto \begin{pmatrix} c_1 & \ldots & c_k \end{pmatrix}_* [\{1-z_1, 1-z_2, \ldots, 1-z_n\}] \in K_n^M(k(\mathbb{G}_m))/I$$

with $c_i = \gamma_i \ldots \gamma_1 e_1$. When we wish to use the Steinberg quotient, however, we will generally work directly with the modular symbol, as it retains more information than the associated cohomology class or cocycle.

Remark 3.3. As suggested in [SV, §5], the cocycles of the form $\Theta(n)$ come from equivariant motivic polylogarithms for the action of $\mathrm{GL}_n(\mathbb{Z})$ on the group scheme $\mathbb{G}_m^n$. The argument of Sharifi-Venkatesh essentially proves this for $n = 2$ by realizing their chain complex in a Gersten complex computing motivic cohomology; our realization map could similarly be extended to map from the whole spherical chain complex to a Gersten complex as well. However, for general n, the Gersten complex does not necessarily compute motivic cohomology, so we would need instead a symbol complex with fewer relations, in order to map to the Bloch cycle complex (which *does* compute motivic cohomology); such a complex can in fact be constructed using matroids. We omit these arguments from this article due to their considerable length and technical overhead, and irrelevance to our main results; however, we will use the matroid approach and prove the relationship with polylogarithms in the sequel to this article, in the elliptic setting (which degenerates at the cusps to the setting of the present article). In that setting, having the formalism of equivariant motivic polylogarithms is useful for comparison reasons.

However, for the *regulator* of the motivic cocycle, we have included the proof the comparison with an equivariant polylogarithm class in de Rham/coherent cohomology, in Appendix A, as this requires less technical overhead. The flavor of the argument would be exactly the same in motivic cohomology.

3.2. **Specialization at torsion sections and Sharifi maps.** For briefness, we have only been working over the generic point until now, but to construct and analyze the properties of the maps in Theorem 1.2, we will need to consider specializations of our cocycles to torsion points: let $\Gamma \subset \mathrm{GL}_n(\mathbb{Q})$ be any subgroup. The proof of Theorem 3.1 applies identically to show that the realization map factors, as a Γ-map, through $H^n(U_\Gamma, \mathbb{Z}(n))^{(0)}$, where U_Γ is defined by

$$U_\Gamma := \varinjlim_H \mathbb{G}_m^n - H \tag{3.9}$$

and the direct limit ranges over finite subarrangements of the hyperplane arrangement which is the Γ-orbit of

$$\mathbb{G}_m^{n-1} \subset \mathbb{G}_m^n$$

embedded as the kernel of $1 - z_n$. For $n \geqslant 3$, the norm residue isomorphism theorem and cohomological dimension arguments show that all pulled back cocycles by torsion points are trivial; thus, from here on, we work only in the case $n = 2$. (For $n = 1$, we get classical cyclotomic N-units, for pullback by N-torsion.)

Remark 3.4. Though the torsion specializations for $n > 2$ are not expected to be interesting, we expect that the pre-specializations cocycles valued in cohomology of open subschemes of $\mathbb{G}_m^n$ should still contain interesting information about L-values, following the philosophy of the "Eisenstein symbol" approach to polylogarithm classes [Beil]. The non-properness of $\mathbb{G}_m$ makes these considerations technically daunting, however; we will thus treat this kind of perspective in more depth in the sequel (on the elliptic case). Nevertheless, we would be very interested in an approach using only the toric cocycle.

Suppose now that $\Gamma = \Gamma_0(N)$ fixes the line generated by a torsion section $s_N : \operatorname{Spec}\mathbb{Q}(\mu_N) \to \mathbb{G}_m^2$ which is $(1, \zeta_N)$ in the coordinates z_1, z_2.

While $\Gamma_0(N)$ does not fix s_N, since it fixes the line generated by x, we can define a homomorphism

$$\sigma : \Gamma_0(N) \to (\mathbb{Z}/N\mathbb{Z})^\times$$

by the rule $\gamma s_N = \sigma(\gamma) s_N$, with kernel the index-$\varphi(N)$ subgroup $\Gamma_1(N)$ fixing s_N. We can then define an action of $\Gamma_0(N)$ on $\mathbb{Q}(\mu_N)$ by

$$\gamma \mapsto ([\sigma(\gamma^{-1})] : \zeta_N \mapsto \zeta_N^{\sigma(\gamma^{-1})}),$$

which yields also pullback maps on motivic cohomology. Now, the pullback

$$s_N^* : H^2(U_\Gamma, \mathbb{Z}(2))^{(0)} \to H^2(\mathbb{Q}(\mu_N), \mathbb{Z}(2))$$

is a priori only $\Gamma_1(N)$-equivariant, but with our newly defined action of $\Gamma_0(N)$ on the right-hand side, one can check (cf. [SV, §4]) that it is actually $\Gamma_0(N)$-equivariant. We thus get a corresponding specialization of our Eisenstein cocycle

$$s_N^* \Theta(2) \in H^1(\Gamma_0(N), H^2(\mathbb{Q}(\mu_N), \mathbb{Z}(2)))$$

In fact, the localization sequence (2.1) for

$$\bigoplus_{\mathfrak{p} \nmid N} \mathbb{Z}[\mu_N]/\mathfrak{p} \hookrightarrow \mathbb{Z}[\mu_N, N^{-1}]$$

yields a left-exact sequence

$$H^2(\mathbb{Z}[\mu_n, N^{-1}], \mathbb{Z}(2)) \hookrightarrow K_2^M(\mathbb{Q}(\mu_N)) \xrightarrow{\partial} \bigoplus_{\mathfrak{p} \nmid N} K_1^M(\mathbb{Z}[\mu_N]/\mathfrak{p})$$

where the last arrow is are the tame symbols which vanish on integral-at-$\mathfrak{p}$ elements. The injectivity is because the Milnor K-theory of finite fields vanishes above degree 1, as one can always express 1 as the sum of quadratic residues. Taking the restriction of the action via σ of $\Gamma_0(N)$ to the cohomology of $\mathbb{Z}[\zeta_N, N^{-1}]$, we see that our pullbacks are actually valued in the integral-away-from-N submodule, i.e. we actually have a cocycle

$$s_N^* \Theta(2) \in H^1(\Gamma_0(N), H^n(\mathbb{Z}[\zeta_N, N^{-1}], \mathbb{Z}[\tfrac{1}{2}](2)))$$

This argument will be applied implicitly also to everything that follows, as well in the next subsection.

One can equally make a "Steinberg" version of this construction: if we let $\mathrm{St}(2)^\circ$ be the $\Gamma_0(N)$-invariant submodule of $\mathrm{St}(2)$ generated by lines not reducing to $[0:1]$ modulo N (i.e. cusps not in the $\Gamma_0(N)$-orbit of $[0:1]$), then the argument of section

3.1.1 affords us a map

$$\Theta^{MS}(2)_{(N)} : \mathrm{St}(2)^\circ \to \varinjlim_{H} H^2(\mathbb{G}_m^2 - H, \mathbb{Z}(2))/I_{(N)}$$

where $I_{(N)}$ is module of relations spanned by symbols in the orbit of

$$\{-z_1, 1 - z_2\} \tag{3.10}$$

under matrices of the form $L = \begin{pmatrix} \ell_1 & \ell_2 \end{pmatrix}$, for ℓ_1, ℓ_2 cusps not in the $\Gamma_0(N)$-orbit of $[0 : 1]$, as above. This affords us a modular symbol

$$s_N^* \Theta^{MS}(2)_{(N)} : [\ell_1] \wedge [\ell_2] \mapsto s_N^* \begin{pmatrix} \ell_1 & \ell_2 \end{pmatrix}_* \{1 - z_1, 1 - z_2\} \in H^2(\mathbb{Z}[\mu_N, N^{-1}], \mathbb{Z}[\tfrac{1}{2}](2))/\text{extra relations from } I_{(N)}$$

After pullback, the module of relations $I_{(N)}$ consists of elements of the form

$$\{-\zeta_N^i, -\zeta_N^j\}, \{-\zeta_N^i, 1 - \zeta_N^j\}.$$

The former relations $\{-\zeta_N^i, -\zeta_N^j\} = 0$ are all true up to 2-torsion by alternation of the Steinberg symbol. The latter relations vanish upon taking the projection onto the plus part

$$(\bullet)_+ : \{x, y\} \mapsto \frac{1}{2}(\{x, y\} + \{\overline{x}, \overline{y}\}),$$

since, with 2 inverted, we have

$$2\{-\zeta_N^i, 1 - \zeta_N^j\}_+ = \{\zeta_N^{-i}, 1 - \zeta_N^{-j}\} + \{\zeta_N^i, 1 - \zeta_N^j\} = \{\zeta_N^i, -\zeta_N^j\} = 0.$$

If we restrict now to the subgroup $\Gamma_1(N)$ fixing $s_N := (1, \zeta_N)$, we thus obtain a specialization

$$(\Pi_N^\circ)_+ := (s_N^* \Theta^{MS}(2)_{(N)})_+ : \mathrm{St}(2)^\circ \to H^2(\mathbb{Z}[\zeta_N, N^{-1}], \mathbb{Z}[\tfrac{1}{2}](2))_+$$

which is a $\Gamma_1(N)$-invariant modular symbol with the trivial action on the target. However, if $N = p^k$, the relations $\{-\zeta_N^i, 1 - \zeta_N^j\}$ are zero before projection: the cyclotomic distribution relation

$$1 - \zeta_{p^k}^{pt} = \prod_{i=0}^{p-1} 1 - \zeta_{p^k}^{t + p^{k-1} i}$$

means we can assume $(j, p) = 1$, in which case

$$\{\zeta_N^i, 1 - \zeta_N^j\} = k\{\zeta_N^j, 1 - \zeta_N^j\} = 0$$

by the Steinberg relation $\{x, 1 - x\} = 0$, where here $kj \equiv i \pmod{p^k}$. Thus in this case, we have a map

$$\Pi_N^\circ := (s_N^* \Theta^{MS}(2)_{(N)}) : \mathrm{St}(2)^\circ \to H^2(\mathbb{Z}[\zeta_N, N^{-1}], \mathbb{Z}[\tfrac{1}{2}](2))$$

In the remainder of this section, for brevity of notation we will write everything without the $+$ projection, with the understanding that we always mean the $+$ part except for when $N = p^s$.

Consider now a *unimodular* symbol $[\ell_1] \wedge [\ell_2]$, i.e. so that one for which the associated matrix formed from integer generators of these lines satisfies

$$\det \begin{pmatrix} \ell_1 & \ell_2 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}).$$

The map of [SV, §4] is defined in terms of these, writing all symbols as sums thereof via their "connecting sequences" (which we avoid due to using pushforwards). To compare our map with theirs, we have that if $\ell_1, \ell_2 \neq [0 : 1]$ (meaning that that s_N is not in the polar locus and the pullback is well-defined),

$$s_N^*\Theta^{MS}(2)_{(N)}([\ell_1]\wedge[\ell_2])) = \left(\begin{pmatrix}\ell_1 & \ell_2\end{pmatrix}^{-1} s_N\right)^* \{1-z_1, 1-z_2\} = \{1-\zeta_N^{-v}, 1-\zeta_N^{u}\} \in H^2(\mathbb{Z}[\zeta_N, N^{-1}], \mathbb{Z}[\tfrac{1}{2}](2))$$

where (u,v) is the top row of the matrix $\begin{pmatrix}\ell_1 & \ell_2\end{pmatrix}$. On symbols disjoint from $[0:1]$, therefore, up to sign and convention of basis, our specialization coincides with the Eisenstein cocycle defined in [SV, §4], as they agree on unimodular generators. (In loc. cit., (u,v) is the bottom row of the analogous matrix; this corresponds to swapping the role of the standard basis lines e_1 and e_2.)

Remark 3.5. The above calculation shows that the relations of the form $\{-\zeta_N^i, 1-\zeta_N^j\}$, where ζ_N^i and ζ_N^j have orders divisible by distinct primes, also do not appear if we omit symbols in the Steinberg module containing lines $[a:b]$ with a divisible by primes dividing N; thus, the full (non-plus part) Π_N° can also be constructed (and shown to be Hecke equivariant, etc., as below) if we are willing to restrict our set of symbols: this corresponds to restricting the set of cusps to those not in the $\Gamma_0(p)$-orbit of ∞ for any prime $p|N$ (in the language of the following section; see below). This remark explains how the calculations of [SV, §4] result in a parabolic cocycle while only inverting 2 in the coefficients, as they only work with homology of the closed curve.

3.2.1. *Hecke operators and modular symbols.* In this section, we define the Hecke operators and prove Theorem 1.2, for

$$(\Pi_N^\circ)_+ := (s_N^*\Theta^{MS}(2)_{(N)})_+ : \mathrm{St}(2)^\circ_{\Gamma_1(N)} \to H^2(\mathbb{Z}[\mu_N, N^{-1}], \mathbb{Z}[\tfrac{1}{2}])_+$$

though as discussed previously, we will omit the $+$ signs for ease of notation, with the understanding that this is only *actually* allowed if $N = p^s$ (or by omitting cusps, as in the remark at the end of last section). Let us explain the notation (also used in Theorem 1.2). Write $X_1(N)$ for the compactified modular curve of level $\Gamma_1(N)$, and $C_1(N)$ for its set of cusps. Then there is a natural identification between coinvariants of the Steinberg module and the Borel-Moore homology:

$$\mathrm{St}(2)_{\Gamma_1(N)} \xrightarrow{\sim} H_1(X_1(N), C_1(N), \mathbb{Z})$$

sending $[\ell_1]\wedge[\ell_2]$ to the image of the geodesic path usually denoted $\{\ell_1, \ell_2\}$ between the cusps corresponding to $\ell_1, \ell_2 \in \mathbb{P}^1(\mathbb{Q})$ under the uniformization of $Y_1(N) := X_1(N) - C_1(N)$ by the complex upper half-plane; similarly, the co-invariants of Steinberg symbols $\mathrm{St}(2)^\circ_{\Gamma_1(N)}$ disjoint from $\Gamma_1(N)[0:1]$ can be identified with the homology relative to the *restricted* cusps $C_1(N)^\circ$ disjoint from $\Gamma_0(N)\cdot\infty$ [AR]. Hence, modular symbols (respectively, modular symbols restricted away from ∞, modular symbols restricted to a single cusp) valued in a trivial $\Gamma_1(N)$-module can be identified with maps from the homology of $(X_1(N), C_1(N))$ (respectively, $(X_1(N), C_1(N)^\circ)$). More generally, we can take any $\Gamma_1(N)$-invariant set of cusps, and obtain as coinvariants the homology relative to only those cusps.

Now we define Hecke operators on such symbols. A useful notion will be the *adjugate* of an invertible matrix

$$\mathrm{adj}(M) := (\det M)M^{-1}.$$

Note that the adjugate of an integer matrix is an integer matrix, and that the scalar $\det M = M\cdot\mathrm{adj}(M)$ acts trivially on the Steinberg module, and hence on the image of any modular symbol equivariant for a group containing these matrices: in other words, M and its adjugate act as inverses on such elements.

Now, for any double coset

$$\Gamma_1(N)\alpha\Gamma_1(N) \in \Gamma_1(N)\backslash\mathrm{GL}_2(\mathbb{Q})/\Gamma_1(N)$$

with left coset decomposition

$$\Gamma_1(N)\alpha\Gamma_1(N) = \bigcup_i \alpha_i\Gamma_1(N),$$

we can define an associated operator T_α on a modular symbol $c : \mathrm{St}(2)_{\Gamma_1(N)} \to A$ (for an abelian group A) by

$$(T_\alpha c)([\ell_1, \ell_2]) = \sum_i c(\alpha_i^{-1}[\ell_1, \ell_2]).$$

One can check this is well-defined independent of the choice of left coset representatives, and corresponds to the classical Hecke action by correspondences on (co)homology of modular curves (and thereby on modular forms, etc.).

First, if $\alpha_d \in \Gamma_0(N)$ with $\sigma(\alpha) = d$ (i.e. the lower right entry is d modulo N), then from previous discussion, the double coset T_α depends only on d; we write $T_\alpha = \langle d \rangle$ and call it a diamond operator.

Next, for any prime $p \nmid N$, we have the double coset operator

$$T_p := \Gamma_1(N) \begin{pmatrix} p & \\ & 1 \end{pmatrix} \Gamma_1(N)$$

and its dual

$$T_p^* := \Gamma_1(N) \begin{pmatrix} 1 & \\ & p \end{pmatrix} \Gamma_1(N)$$

One computes the relation $T_p^* = \langle p \rangle T_p$. We will consider the following set of coset representatives for T_p^*: by [SV, Theorem 4.3.7], there exists a set of representatives $\alpha_{i,p}$, $0 \leqslant i \leqslant p$, given by

$$\alpha_{i,p} = \begin{pmatrix} 1 & \\ i & p \end{pmatrix}$$

for $i < p$, and

$$\alpha_{p,p} = \begin{pmatrix} p & \\ & 1 \end{pmatrix} \alpha_p.$$

Here, α_p is the representative for the diamond operator $\langle p \rangle$. Note that that considered as maps $\mathbb{Q}^2/\mathbb{Z}^2 \to \mathbb{Q}^2/\mathbb{Z}^2$, the kernel of these $p+1$ matrices are the $p+1$ subgroups of order p, and that $\mathrm{adj}(\alpha_i)$ fixes $(0,1)$ modulo N.

For primes $p|N$, we similarly have

$$U_p := \Gamma_1(N) \begin{pmatrix} p & \\ & 1 \end{pmatrix} \Gamma_1(N).$$

We will consider the left coset decomposition

$$U_p = \bigcup_{i=0}^{p-1} \begin{pmatrix} p & Ni \\ & 1 \end{pmatrix} \Gamma_1(N).$$

This operator also has a dual, which we ignore (since our symbol will have no simple corresponding equivariance property).

Proof of Theorem 1.2. From previous results, we find that

$$\langle d \rangle s_N^* \Theta^{MS}(2)_{(N)}([\ell_1, \ell_2]) = s_N^* \alpha_*^{-1} \Theta^{MS}(2)_{(N)}([\ell_1, \ell_2]) = [d]^* s_N^* \Theta^{MS}(2)_{(N)}([\ell_1, \ell_2])$$

since $\alpha s_N = [d] s_N$.

We conclude that $\langle d \rangle \Pi_N^\circ = [d]^* \Pi_N^\circ$, as desired.

For the operator T_p^*, $p \nmid N$, we have

$$T_p^* s_N^* \Theta^{MS}(2)_{(N)}([\ell_1, \ell_2]) = \sum_{i=0}^{p} s_N^* \Theta^{MS}(2)_{(N)}(\alpha_{i,p}^{-1}[\ell_1, \ell_2]) \tag{3.11}$$

$$= \sum_{i=0}^{p} s_N^* (\mathrm{adj}(\alpha_{i,p}))_* \Theta^{MS}(2)_{(N)}([\ell_1, \ell_2]) \tag{3.12}$$

$$= \sum_{i=0}^{p} s_N^* (\mathrm{adj}(\alpha_{i,p}))^* (\mathrm{adj}(\alpha_{i,p}))_* \Theta^{MS}(2)_{(N)}([\ell_1, \ell_2]) \tag{3.13}$$

Now observe that for matrix $M : \mathbb{G}_m^2 \to \mathbb{G}_m^2$, the correspondence $M^* M_*$ is precisely consists of $(x, x \cdot \ker M)$ where $\cdot \ker M$ means translation by this subgroup. Thus, since the kernel of the various $\alpha_{i,p}$ are precisely the $p+1$ lines of order p torsion, we have the equality of correspondences

$$\sum_{i=0}^{p} \alpha_{i,p}^* (\alpha_{i,p})_* = (x, px + x \cdot \ker [p]) \subset \mathbb{G}_m^2$$

since the union of all p-torsion lines is precisely the p-torsion, except the identity is counted once per line. These correspondences also all preserve the open set $U_{\Gamma_1(N)}$, and hence can be applied to the values of the spread-out cocycle $\Theta^{MS}(2)_{(N)}$. Since $\Theta^{MS}(2)_{(N)}([\ell_1, \ell_2])$ is $[p]_*$-invariant, we finally obtain

$$T_p^* s_N^* \Theta^{MS}(2)_{(N)}([\ell_1, \ell_2]) = s_N^* (p + [p]^*) \Theta^{MS}(2)_{(N)}([\ell_1, \ell_2]) \tag{3.14}$$

$$= p s_N^* \langle p \rangle s_N^* \Theta^{MS}(2)_{(N)}([\ell_1, \ell_2]) \tag{3.15}$$

so that $T_p^* \Pi_n^\circ = (p + \langle p \rangle) \Pi_n^\circ$, as desired.

Finally, for the operator U_p, $p | N$, we write:

$$U_p s_N^* \Theta^{MS}(2)_{(N)}([\ell_1, \ell_2]) = \sum_{i=0}^{p-1} s_N^* \begin{pmatrix} 1 & -Ni \\ & p \end{pmatrix}_* \Theta^{MS}(2)_{(N)}([\ell_1, \ell_2]) \tag{3.16}$$

$$= \sum_{i=0}^{p-1} s_N^* \begin{pmatrix} 1 & -\frac{N}{p} i \\ & 1 \end{pmatrix}_* \begin{pmatrix} 1 & \\ & p \end{pmatrix}_* \Theta^{MS}(2)_{(N)}([\ell_1, \ell_2]) \tag{3.17}$$

$$= \sum_{i=0}^{p-1} (\zeta_p^i, \zeta_N)^* \begin{pmatrix} 1 & \\ & p \end{pmatrix}_* \Theta^{MS}(2)_{(N)}([\ell_1, \ell_2]) \tag{3.18}$$

$$= s_N^* \begin{pmatrix} p & \\ & p \end{pmatrix}_* \Theta^{MS}(2)_{(N)}([\ell_1, \ell_2]) \tag{3.19}$$

$$= s_N^* \Theta^{MS}(2)_{(N)}([\ell_1, \ell_2]) \tag{3.20}$$

from which $U_p = 1$ on Π_N° follows. □

We also furnish a new proof of the N-integrality of Π_N° (or $(\Pi_N^\circ)_+$, depending on N) when restricted to the homology of the compact curve (compare [SV, Lemma 4.2.7], [FK, Lemma 3.3.11]). For this, we make the observation that the homology of the closed modular curve $X_1(N)$ is a submodule

$$H_1(X_1(N)) \hookrightarrow H_1(X_1(N), C_1(N))$$

described in terms of the Steinberg module as being generated by symbols of the form $\gamma_1[1:0] \wedge \gamma_2[1:0]$ for $\gamma_1, \gamma_2 \in \Gamma_1(N)$, i.e. geodesic paths between cusps in the orbit of the zero cusp (or, equivalently, any given fixed cusp).

Theorem 3.6. *The restriction of* Π_N° *(respectively* $(\Pi_N^\circ)_+$*, when* N *has distinct prime divisors) to* $H_1(X_1(N))$ *takes values in the submodule*

$$H^2(\mathbb{Z}[\mu_N], \mathbb{Z}[\tfrac{1}{2}](2)) \hookrightarrow H^2(\mathbb{Z}[\mu_N, N^{-1}], \mathbb{Z}[\tfrac{1}{2}](2))$$

(respectively with $+$ *parts).*

Proof. Again, we will write the proof without $+$ projections, for brevity. By Proposition 2.1, the $H^2(\mathbb{Z}[\mu_N], \mathbb{Z}[\frac{1}{2}](2))$ is the submodule of $H^2(\mathbb{Q}(\mu_N), \mathbb{Z}[\frac{1}{2}](2))$ on which the various tame symbols

$$H^2(\mathbb{Q}(\mu_N), \mathbb{Z}[\tfrac{1}{2}](2)) \to H^1(\mathbb{Z}[\mu_N]/\mathfrak{p}, \mathbb{Z}[\tfrac{1}{2}](2))$$

vanish, for *all* primes $\mathfrak{p}$. Let

$$L = \begin{pmatrix} \ell_1 & \ell_2 \end{pmatrix}$$

be the pushforward matrix associated to a Steinberg symbol in $\mathrm{St}(2)^\circ$. We have the following pullback/pushforward functoriality of tame symbols [Lev2]:

$$\begin{array}{ccc} H^2(U_{\Gamma_1(N)}, \mathbb{Z}[\frac{1}{2}](2)) & \xrightarrow{\partial} & \bigoplus_D H^1(D, \mathbb{Z}[\frac{1}{2}](2)) \\ \downarrow{\scriptstyle L_*} & & \downarrow{\scriptstyle L_*} \\ H^2(U_{\Gamma_1(N)}, \mathbb{Z}[\frac{1}{2}](2)) & \xrightarrow{\partial} & \bigoplus_D H^1(D, \mathbb{Z}[\frac{1}{2}](2)) \\ \downarrow{\scriptstyle s_N^*} & & \downarrow{\scriptstyle s_N^*} \\ H^2(\mathbb{Q}(\zeta_N), \mathbb{Z}[\frac{1}{2}](2)) & \xrightarrow{\partial} & \bigoplus_{\mathfrak{p}} H^1(\mathbb{Z}[\zeta_N]/\mathfrak{p}, \mathbb{Z}[\frac{1}{2}](2)) \end{array} \tag{3.21}$$

where the direct sums range over irreducible components of the codimension-1 locus $\mathbb{G}_m - U_{\Gamma_1(N)}$ and closed points of $\operatorname{Spec} \mathbb{Z}[\zeta_N]$ respectively, indexed by convention and familiarity by "D" for divisor (in $\mathbb{G}_m^2$) for the geometrically-flavored $U_{\Gamma_1(N)}$, and by the associated prime ideal $\mathfrak{p}$ for the cyclotomic number ring. In the lower right vertical arrow, s_N^* is zero by convention if s_N fails to properly intersect D.

Now, the value $\Pi_N^\circ([\ell_1] \wedge [\ell_2])$ is the image in the bottom left group of $\{1 - z_1, 1 - z_2\}$ in the top left group, whose tame symbol is

$$\{1 - z_2\}_{z_1 = 0} - \{1 - z_1\}_{z_2 = 0}.$$

Let the coordinates of (integral generators of) the lines ℓ_1, ℓ_2 be considered as maps $(\ell_1), (\ell_2) : \mathbb{G}_m \mapsto \mathbb{G}_m^2$. Then commutativity of the above diagram implies that

$$\partial \Pi_N^\circ([\ell_1] \wedge [\ell_2]) = s_N^* L_* \{1 - z_2\}_{z_1 = 0} - \{1 - z_1\}_{z_2 = 0} \tag{3.22}$$

$$= (1, \zeta_N)^* [(\ell_1)_* \{1 - z\} - (\ell_2)_* \{1 - z\}] \tag{3.23}$$

which vanishes when $\ell_1 = \gamma \ell_2$ for $\gamma \in \Gamma_1(N)$, as in that case $(\ell_1) = \gamma \circ (\ell_2)$ as maps, and we have

$$(1, \zeta_N)^* \gamma_* = (\gamma^{-1}(1, \zeta_N))^* = (1, \zeta_N)^*$$

as maps. ☐

Remark 3.7. A certain level compatibility for these Eisenstein cocycles was proven relative to more restrictive sets of cusps in [LcW, Theorem 1.1], and asked whether a version existed relative to larger sets of cusps. We expect that the methods of that article applied in our formalism furnish a proof of this extended level compatibility. We do not go into the details as the main purpose of the authors of that article was to prove Hecke equivariance results, which we have established by other means.

4. THE DE RHAM COCYCLES AND APPLICATIONS

In this section, we pass from our motivic/K-theory-valued class to one valued in differential forms, from which we will be able to extract actual numbers (or distributions) related to L-values of totally real fields.

There is a regulator map [LW, §2.1.5]

$$(d\log)^{\wedge n} : H^n(X, \mathbb{Z}(n)) \to \Omega^n_X, \{u_1, \ldots, u_n\} \mapsto d\log u_1 \wedge \ldots \wedge d\log u_n$$

which exists for any scheme X, functorial for pullbacks and pushforwards, to turn our motivic-valued cocycles into differential form-valued cocycles. We deduce from our motivic cocycle

$$(d\log)^{\wedge n}_* \Theta(n) \in H^{n-1}(\mathrm{GL}_n(\mathbb{Q}), (\Omega^n_{k(\mathbb{G}^n_m)})^{(0)} / \langle (z_1 \ldots z_n)^{-1} dz_1 \wedge \ldots \wedge dz_n \rangle)$$

which can be represented by a homogeneous cocycle which sends

$$(\gamma_0, \ldots, \gamma_{n-1}) \mapsto \begin{pmatrix} \gamma_0 e_1 & \ldots \gamma_{n-1} e_1 \end{pmatrix}_* \frac{(-1)^n}{(1 - z_1) \ldots (1 - z_n)} dz_1 \wedge \ldots \wedge dz_n$$

whenever the first columns of $\gamma_0, \ldots, \gamma_{n-1}$ are independent (or an analogous condition if we replace e_1 with any rational ray). If we write $z_i = \exp(2\pi i t_i)$ for $1 \leqslant i \leqslant n$, then we have

$$dz_1 \wedge \ldots \wedge dz_n = z_1 z_2 \ldots z_n dt_1 \wedge \ldots \wedge dt_n.$$

The n-form $dt_1 \wedge \ldots \wedge dt_n$ transforms by the character $\det$ under pullback by $\mathrm{GL}_n(\mathbb{Q})$, so

$$\omega \mapsto \frac{\omega}{dt_1 \wedge \ldots \wedge dt_n}$$

furnishes a $\mathrm{GL}_n(\mathbb{Q})$-equivariant isomorphism

$$\iota_t : \Omega^n_{k(\mathbb{G}^n_m)} / \langle dz_1 \wedge \ldots \wedge dz_n \rangle \to (\mathcal{M}_{\mathbb{G}^n_m} / \langle 1 \rangle)(-\det)$$

by the push-pull formula

$$\gamma_*(f\, dt_1 \wedge \ldots \wedge dt_n) = \frac{1}{\det \gamma} \gamma_*(f\, \gamma^* dt_1 \wedge \ldots \wedge dt_n) = \frac{1}{\det \gamma} \gamma_*(f)\, dt_1 \wedge \ldots \wedge dt_n$$

Here, $\mathcal{M}_{\mathbb{G}^n_m}(-\det)$ denotes the meromorphic functions with their pushforward action twisted by the character $\det^{-1}$. We then define our differential Eisenstein cocycle as

$$\Theta^{dR}(n) := (\iota_t)_* \circ (d\log)^{\wedge n}_* \Theta(n) \in H^{n-1}(\mathrm{GL}_n(\mathbb{Q}), (\mathcal{M}_{\mathbb{G}^n_m} / \langle 1 \rangle (-\det))^{(0)})$$

which can be represented by a homogeneous cocycle sending

$$(\gamma_0, \ldots, \gamma_{n-1}) \mapsto (\det M)^{-1} M_* \frac{z_1 z_2 \ldots z_n}{(z_1 - 1) \ldots (z_n - 1)}$$

for

$$M = \begin{pmatrix} \gamma_0 e_1 & \ldots \gamma_{n-1} e_1 \end{pmatrix}$$

whenever the first columns of $\gamma_0, \ldots, \gamma_{n-1}$ are independent. Here, the superscript (0) means trace-fixed *with* the determinant twist; i.e. $[a]_* f = a^n f$ for $a \in \mathbb{N}$.

In fact, if we restrict to $\mathrm{GL}_n(\mathbb{Z}[S^{-1}])$ for some set of primes S, Corollary 2.11 and (2.14) tell us that associated to a lift ϕ of the Euler class we obtain

$$\Theta^{dR,\phi}(n) \in d_{n,S}^{-1} \cdot H^{n-1}(\mathrm{GL}_n(\mathbb{Z}[S^{-1}]), \mathcal{M}_{\mathbb{G}_m^n}(-\det)^{(0)}).$$

For our application to L-values of totally real fields, we will be interested in the case $S = \varnothing$, for which d_n is the denominator of the halved Bernoulli number $\frac{B_n}{2}$. For this reason, we will also restrict to the subgroup $\mathrm{GL}_n(\mathbb{Z})$ in all that follows, though many of the same statements also hold with essentially the same proofs for S-arithmetic groups.

It will be useful to introduce the shorthand

$$\tilde{r} := \iota_t \circ d\log^{\wedge n} \circ \tilde{\rho} : \tilde{C}(n) \to \mathcal{M}_{\mathbb{G}_m^n}(-\det)^{(0)}$$

for the realization map from rational spherical chains to meromorphic functions.

4.1. **Some transforms of the cocycle.**

4.1.1. *Stabilizations.* To obtain relations to totally real L-values, we will need certain combinations of torsion-section translates of our de Rham cocycles. In this section, we introduce the requisite notation, and make some basic observations about the resulting stabilizations when some degree-zero properties are satisfied.

Over $\overline{\mathbb{Q}}$, the torsion of $\mathbb{G}_m^n$ is the nth power of the roots of unity μ_∞^n; given $x \in \mu_\infty^n$, we will write

$$t_x : \mathbb{G}_m^n \mapsto \mathbb{G}_m^n$$

for the translation map which is multplication by the *inverse* x^{-1}. More generally, via the complex uniformization $\mathbb{G}_m^n \cong \mathbb{C}^n/\mathbb{Z}^n$, we can identify torsion sections with elements of $\mathbb{Q}^n/\mathbb{Z}^n$. Then to any Schwartz function $\varphi \in \mathcal{S}(\mathbb{Q}^n/\mathbb{Z}^n)$ with period $\mathbb{Z}^n$ (in other words, finitely supported functions on $\mathbb{Q}^n/\mathbb{Z}^n$), we define an associated pullback operator on $\mathcal{M}_{\mathbb{G}_m^n}(-\det)$ by

$$t_\varphi^* := \sum_{x \in \mathbb{Q}^n/\mathbb{Z}^n} \varphi(x) \cdot t_x^*.$$

The group $\mathrm{GL}_n(\mathbb{Z})$ acts on $\mathbb{Q}^n/\mathbb{Z}^n$ by the standard left action, and for this action, we have the equivariance property

$$\gamma_* t_\varphi^* f = t_{\varphi \circ \gamma^{-1}}^* \gamma_* f$$

for any $f \in \mathcal{M}_{\mathbb{G}_m^n}$.

We observe that if φ is Γ-invariant, for an arithmetic subgroup $\Gamma \subset \mathrm{GL}_n(\mathbb{Z})$, then $t_\varphi^* \Theta^{dR}(n)$ is a cohomology class for Γ; this applies to all variants of this construction we have seen as well, and to the cocycle representatives we have written down. In particular, associated to a Δ-extension E and an Euler class lift ϕ, we have a cocycle representative

$$t_\varphi^* \theta_{E,\phi}^{dR}(n)$$

If φ sums to zero, t_φ^* annihilates the Euler class, so ϕ can be taken to be zero and omitted from the notation. Furthermore, we recall that the ambiguity in the choice of a Δ-extension is only on degenerate "wedge"-like simplices $\Delta^E(r_1, \ldots, r_n)$ with a linear dependence among the rays. For such a simplex, suppose its rays span a proper subspace $V \subset \mathbb{Q}^n$. Notice that the realization of the standard wedge $\mathbb{Q}e_1 \oplus \ldots \oplus \mathbb{Q}e_k \oplus \mathbb{Q}^+ e_{k+1} \oplus \ldots \mathbb{Q}^+ e_n$ is

$$\frac{z_{k+1} \ldots z_k}{(z_{k+1} - 1) \ldots (z_n - 1)},$$

which is unaffected by translation by elements in V, meaning its φ-weighted sum over any V-coset is just the total mass of φ on that coset. By $\mathrm{GL}_n(\mathbb{Q})$-equivariance, this implies that in general, if φ sums to zero considered as a function on $(\mathbb{Q}^n/\mathbb{Z}^n)/V(\mathbb{Q}/\mathbb{Z})$, then $t_\varphi^* \tilde{r}_* \Delta^E(r_1, \ldots, r_n) = 0$ as well regardless of the choice of Δ-extension.

For instance, if φ sums to zero on *every* subspace spanned by Γ-orbits of e_1, then $t_\varphi^* \theta^{dR,E}(n)$ is also independent of E (which can thus be omitted from the notation in this case), and simply vanishes on *all* linearly dependent tuples of rays arising from tuples of matrices in Γ. This phemomenon, wherein realizations of "wedge" classes disappear upon stabilization, will be important in our application to Shintani domains/L-values of totally real fields. It also arises in the following comparison:

Example 4.1. The analytic "cocycle multiplicatif" of [BCG2] is a homogeneous $(n-1)$-cocycle

$$\mathbf{S}^*_{\mathrm{mult}}[\varphi_f] : \Gamma^n \to \mathcal{M}_{\mathbb{G}_m^n},$$

for the *right* pullback action on the coefficients, where $\varphi \in \mathcal{S}(\mathbb{Q}^n/\mathbb{Z}^n)$ is Γ-invariant for some arithmetic subgroup $\Gamma \subset \mathrm{SL}_n(\mathbb{Z})$, and sums to zero on any subspace containing the line corresponding to the first standard basis element e_1.

In [BCG2, Proposition 8.8], the value of $\mathbf{S}^*_{\mathrm{mult}}[\varphi]$ on a tuple $(\gamma_0, \ldots, \gamma_{n-1})$ is computed as

$$\sum_{v \in \mathbb{Q}^n/\mathbb{Z}^n} \varphi(v) \sum_{\substack{\xi \in \mathbb{Q}^n/\mathbb{Z}^n \\ h\xi \equiv v \pmod{\mathbb{Z}^n}}} \frac{d\ell_1 \wedge \ldots \wedge d\ell_n}{(e^{2\pi i(\ell_1 - \xi_1)} - 1) \ldots (e^{2\pi i(\ell_n - \xi_n)} - 1)}$$

whenever

$$h := \begin{pmatrix} \gamma_0^{-1} e_1 & \ldots & \gamma_{n-1}^{-1} e_1 \end{pmatrix}$$

has linearly independent columns and $h^* \ell_j = e_j^\vee$. We find that upon restriction of our cocycle to Γ, we have

$$\mathbf{S}^*_{\mathrm{mult}}[\varphi](\gamma_0^{-1}, \ldots, \gamma_{n-1}^{-1}) = t^*_{\varphi_f} \theta^{dR}(n)(\gamma_0, \ldots, \gamma_{n-1}) \tag{4.1}$$

where here we use

$$\frac{1}{x-1} = \frac{x}{x-1} - 1$$

and then the degree-zero property of φ to see that the contributions of the constants -1 cancel out. As remarked above, our cocycle representative $t_\varphi^* \theta^{dR}(n)(\gamma_0, \ldots, \gamma_{n-1})$ equals zero independently of a choice of Δ-extension when h is not full rank. Note that the presence of inverses makes sense, since our cocycle is a *left* cocycle while theirs is a right cocycle. This reflects their systematic use of pullback actions rather than pushforwards; since we are working with subgroups of $\mathrm{SL}_n(\mathbb{Z})$, the pushforward coincides with the pullback of the inverse. In fact, this comparison holds in more generality without need for such stringent degree-zero assumptions: [BCG2, Theorem 1.7] identifies $\mathbf{S}^*_{\mathrm{mult}}[\varphi_f]$ with the image under an edge map of a certain *equivariant polylogarithm* class for $\mathbb{G}_m^n$, up to some controlled ambiguity; we prove the same for our cocycle in Appendix A.

4.1.2. *p-power distributions.* We now discuss the *specializations* of elements of $\mathcal{M}_{\mathbb{G}_m^n}(-\det)^{(0)}$ at torsion points of $\mathbb{G}_m^n$, and in particular specializations to distributions over all p-power torsion at once (which will furnish the link to p-adic L-functions).

A general meromorphic function cannot be specialized at arbitrary torsion points, since there may be poles, so we will need to stabilize. For simplicity, we will from this point on fix one prime p, and will focus on specializations at Np^∞-torsion. Let $A_p \hookrightarrow \mathcal{M}_{\mathbb{G}_m^n}$ be the meromorphic functions on $\mathbb{G}_m$ which are regular on the open p-adic disc around the identity. Note now that only the matrices in $M_n(\mathbb{Z})$ of determinant invertible in $\mathbb{Z}[1/p]$ may act by pushforward on this space: in analogy with preceding notation, we write $A_p(-\det)^{(0)}$ for the $[p]_*$-fixed vectors in the corresponding representation of $\mathrm{GL}_n(\mathbb{Z})$, noting that $[p]_*$ is the only remaining trace allowed, and the determinant is only a sign.

Example 4.2. As an example, for any torsion point x of order $c > 1$ prime to p with all n coordinates nonzero,

$$t_x^* \frac{z_1 \dots z_n}{(1 - z_1) \dots (1 - z_n)} \in A_p.$$

More generally, if $M \in M_n(\mathbb{Z})$ is a matrix, and every torsion section in $M^{-1}x$ has all coordinates nonzero, then

$$t_x^* M_* \frac{z_1 \dots z_n}{(1 - z_1) \dots (1 - z_n)} \in A_p.$$

Elements in A_p can be pulled back by all p-power torsion by definition. Moreover, we have the following formalism in terms of Schwartz functions on $\mathbb{Q}_p^n$:

Proposition 4.3. *There is a unique map*

$$\kappa : A_p \xrightarrow{\sim} \mathbf{D}((\mathbb{Z}_p^n)^\vee, \mathbb{Z}_p)$$

such that

$$\int_{(\mathbb{Z}_p^n)^\vee} e^{2\pi i T(x)}\, d\kappa(f)(T) = f(x)$$

for any $f \in A_p$ and $x \in \mathbb{G}_m^n[p^\infty] = \mathbb{Q}_p^n/\mathbb{Z}_p^n$ (using our fixed identification of $\mathbb{Z}^n$ and the cocharacter lattice of $\mathbb{G}_m^n$, as GL_n-representations).

Proof. This is essentially [Katz, Theorem 1] in slightly different language, except for the minor caveat that Katz's statement involves *measures*, i.e. the continuous dual of continuous functions: since locally constant functions on $\mathbb{Z}_p^n$ are dense in continuous ones, there is a canonical isomorphism

$$\mathbf{D}(\mathbb{Z}_p^n, \mathbb{Z}_p) \cong \mathrm{Measures}(\mathbb{Z}_p^n, \mathbb{Z}_p)$$

given by approximating by Riemann sums; here it is important that we take p-complete coefficients, but otherwise this is an immaterial change of notation.[8] $\square$

We observe the following equivariance properties for the map κ: for $\gamma \in \mathrm{GL}_n(\mathbb{Z}[1/p]) \cap M_n(\mathbb{Z})$ and $f \in A_p(-\det)^{(0)}$, κ restricts to a map

$$A_p(-\det)^{(0)} \to \mathbf{D}^{(0)}((\mathbb{Z}_p^n)^\vee, \mathbb{Z}_p) = \mathbf{D}^{(0)}((\mathbb{Q}_p^n)^\vee, \mathbb{Z}_p)$$

of distributions which are fixed by $[p]_*$, in the sense that U and $[p]U$ have the same measure for all opens $U \subset (\mathbb{Z}_p^n)^\vee$ (where these are extended to $(\mathbb{Q}_p^n)^\vee$ by this property). Indeed, this follows from the calculation that

$$\begin{aligned}
\int_{(\mathbb{Z}_p^n)^\vee} e^{2\pi i T(x)}\, d\kappa([p]_* f)(T) &= \sum_{px'=x} \int_{(\mathbb{Z}_p^n)^\vee} e^{2\pi i T(x')}\, d\kappa(f)(T) \\
&= p^n \int_{p\cdot(\mathbb{Z}_p^n)^\vee} e^{2\pi i ([p]^* T)(x)}\, d\kappa(f)(T) \\
&= p^n \int_{p\cdot(\mathbb{Z}_p^n)^\vee} e^{2\pi i T(x)}\, d([p]_* \kappa(f))(T)
\end{aligned}$$

where the middle equality comes from the corresponding distribution relation on characters.

[8]We are mostly interested in p-adic integration to obtain p-adic L-functions, so for convenience of exposition, we will use p-adic coefficients in all distributions going forward, though many of our classes (for example, ${}_c\Theta^{dR}_{\mathbf{D}_p}(n)$ below) are actually $\mathbb{Z}$-valued when considered only as distributions (i.e. against locally constant test functions). Using the $\mathbb{Z}$-valued distributions would be necessary to imitate the "multiplicative" construction of [RX2], as mentioned in the introduction, for example.

Also, we have the pullback functoriality

$$\kappa(\gamma^* f) = [\gamma^*]_* \kappa(f) \tag{4.2}$$

for any $\gamma \in \mathrm{GL}_n(\mathbb{Z}) \cap M_n(\mathbb{Q})$ of determinant prime to p, following from the computation

$$\int_{(\mathbb{Z}_p^n)^\vee} e^{2\pi i T(x)}\, d\kappa(\gamma^* f)(T) = \int_{(\mathbb{Z}_p^n)^\vee} e^{2\pi i (\gamma^* T)(x)}\, d\kappa(f)(T) = \int_{(\mathbb{Z}_p^n)^\vee} e^{2\pi i T(x)}\, d([\gamma^*]_* \kappa](f)(T)$$

noting that γ^* preserves the lattice $(\mathbb{Z}_p^n)^\vee$ by the determinant condition. Here, $\gamma^* : (\mathbb{Z}^n)^\vee \to (\mathbb{Z}^n)^\vee$ is the usual pullback, and $[\gamma^*]_* := \mathbf{D}((\mathbb{Z}_p^n)^\vee, -) \to \mathbf{D}((\mathbb{Z}_p^n)^\vee, -)$ the induced pushforward on measures.

Observe that the pullback functoriality leads also to pushforward functoriality on $\mathrm{GL}_n(\mathbb{Z})$, as then $\gamma^*\gamma_*$ is the identity on A_p. The map κ, when restricted to (0)-parts, also has a pushforward functoriality for $\mathrm{GL}_n(\mathbb{Z}[1/p])$, but we ignore it as unneeded in this paper, besides the case of $p^{\mathbb{Z}}$ given above. Note that one does not expect pushforward functoriality for matrices with determinant divisible by primes besides p, since Cartier duality does not "see" the prime-to-p torsion.

Going forward, when we only care about the $\mathrm{SL}_n(\mathbb{Z})$-action, we will also sometimes write $A_p^{(n)}$ instead of $A_p(-\det)^{(0)}$, to still indicate that we are considering the functions on which $[p]_*$ acts as p^n (but no longer care about the action of matrices for which the determinant twist is relevant).

4.2. **Shintani generating functions.** We review the method of Shintani for obtaining zeta values of a totally real field F of absolute degree n [Shin], though we formulate things more in the style of [Katz] (with some modernizations of notation); this method is how we will prove interpolation properties connecting our cocycles to L-values.

Let $U := \mathcal{O}_F^\times$, and let U^+ be the totally positive units, i.e. units positive at every real place. We will also write F^+, $\mathcal{O}_F^+$, etc. for analogous constructions.

A *Shintani decomposition* of $F \otimes \mathbb{R}$ is a collection $\mathscr{S}$ of relatively open[9] rational simplicial cones (of various dimensions), each bounded by linearly independent F-rational totally positive rays, such that the U^+-orbit of these cones in $\mathscr{S}$ yield a disjoint partition of $(F \otimes \mathbb{R})^+$: in other words, the union of the cones in such a decomposition constitutes a fundamental ("Shintani") domain for U^+ acting on this orthant.

If $I \subset F$ is any fractional ideal, viewed as a lattice inside $F \otimes \mathbb{R}$, we will be interested in coordinatizations

$$\alpha : F \otimes \mathbb{R} \cong \mathbb{R}^n$$

which restricts to an identification of lattices $\alpha : I \xrightarrow{\sim} \mathbb{Z}^n$. Given a cone (as above) C with bounding rays generated by integral points $x_1, x_2, \ldots, x_k$, write

$$R_C : \mathbb{Z}^n \cap \{c_1 x_1 + \ldots + c_k x_k, \forall i, c_i \in \mathbb{Q}, 0 < c_i \leqslant 1\}$$

for the lattice points in the "unit cube" spanned by the x_i. We also write z_x for the monomial $z_1^{x_1} \ldots z_n^{x_n}$, for any $x \in I \cong \mathbb{Z}^n$.

Then if $\psi : I \to \overline{\mathbb{Q}}^\times$ is any finite order additive character (which we will identify via α with a character of $\mathbb{Z}^n$), identified with a character of $\mathbb{Z}^n$, associated to this data we have a rational "Shintani generating function"

$$f_{\mathscr{S}}(\psi)(z_1, \ldots, z_n) := \sum_{C \in \mathscr{S}} f_C(\psi)(z_1, \ldots, z_n) := \sum_{C \in \mathscr{S}} \sum_{y \in R_C} \frac{\psi(y) z_y}{\prod_x (1 - \psi(x) z_x)} \tag{4.3}$$

[9]Meaning, excluding its lower-dimensional conical faces.

where the product in the denominator is over integral generators for the rays bounding the cone C. Note that here, ψ can be identified with a torsion point in $\mathbb{G}_m^n$; namely, the section $(\zeta_1, \ldots, \zeta_n)$ if $\psi(x_1, \ldots, x_n) = \zeta_1^{x_1} \ldots \zeta_n^{x_n}$. We will also write simply f_C for $f_C(1)$.

The rational function $f_C(\psi)$ is a way to make sense of the naive infinite sum

$$\sum_{x \in C} \psi(x) z_x \tag{4.4}$$

whose meaning is otherwise ambiguous. Note that if $x \in \mathbb{G}_m^n[c]$ is a torsion point, then

$$t_x^* f_C(\xi) = f_C(\psi_x \cdot \xi)$$

for the character $\psi_x : \mathbb{Z}^n \to \overline{\mathbb{Q}}^\times$ given by $(i_1, \ldots, i_n) \mapsto (x_1^{-i_1}, \ldots, x_n^{-i_n})$, i.e. we are implicitly identifying $\mathbb{Z}^n$ with the character lattice of $\mathbb{G}_m^n$.

On these Shintani functions, we also define the differential operator sending

$$D_{\mathbf{N}} : z_x \mapsto \mathbf{N}(x) z_x$$

for any $x \in I$, where we write $\mathbf{N}$ for the field norm of F. The main result of Shintani's approach to L-functions is then the following:

Theorem 4.4. ([Shin, Prop. 1], formulation from [Katz, Theorem 2]) *One can define a meromorphic function by analytically continuing the Dirichlet series*

$$\zeta_I(\psi, s) := \sum_{[\lambda] \in I^+/U^+} \psi(\lambda) \mathbf{N}(\lambda)^{-s}.$$

Then for all $k \geqslant 0$,

$$\zeta_I(\psi, -k) = D_{\mathbf{N}}^k(f_{\mathscr{S}}(\psi))(1, \ldots, 1)$$

for any character ψ nontrivial on each ray bounding any $C \in \mathscr{S}$.

The proof of this theorem is an analytic trick generalizing the proof the functional equation of the Riemann zeta function using an integral representation; what it actually shows is that

$$D_{\mathbf{N}}^k(f_C(\psi))(1, \ldots, 1) = \sum_{\lambda \in C} \psi(\lambda) \mathbf{N}(\lambda)^{-s},$$

so that the fact that the union of $\mathscr{S}$ is a fundamental domain for I^+/U^+ implies the result.

We also note the following relation to p-adic integration of the differential operator $D_{\mathbf{N}}^k$, noted by Katz [Katz, Theorem 1], which will be useful later:

Proposition 4.5. *If $\psi : I^+/U^+ \to \overline{\mathbb{Q}_p}^\times$ is any additive character, we identify ψ with a function $\tilde{\psi}$ on $\mathbb{Z}_p$ via $\psi = \tilde{\psi} \circ \mathbf{N} \circ \alpha^{-1}$. Then for any $k \geqslant 0$, and any $f \in A_p$, we have*

$$\int_{I_p} \psi(t) \mathbf{N}(t)^k \, d\kappa(f)(t) = (D_{\mathbf{N}}^k f)(\psi)$$

where we here identify ψ, as before, with a p-power torsion point of $\mathbb{G}_m^n$, and $I_p := I \otimes \mathbb{Z}_p$.

Note that replacing ψ in all of the above by *any* locally constant function in the p-adic topology, we obtain an analogous result by taking the associated linear combination of additive characters corresponding to the Fourier transform of that function. We will use this more general version without comment.

4.2.1. *Conical duality.* Observe that the functions $f_C(\psi)$ lie in the image of $\Theta^{dR}(n)$ when $\psi = 1$; for general ψ, they are linear combinations of torsion translates of functions in the image. The data of a top-dimensional rational simplicial cone in $\mathbb{R}^n$ is equivalent to a the data of its simplex of intersection with S^{n-1}, i.e. the generators of $\tilde{C}(n)$. One may naively then expect that for an acyclic simplex *qua* cone C, our de Rham realization of $[C]$ as a spherical chain[10] is the Shintani generating function for C; i.e., that $\widetilde{r}([C]) = f_C(1) \in \mathcal{M}_{\mathbb{G}_m^n}$.

However, this is *not* the case: rather, if $C = \Delta(r_1, \ldots, r_n)$ is an acyclic, positively oriented simplex, then

$$f_C(1) = \operatorname{adj}\begin{pmatrix} r_1 & \ldots & r_n \end{pmatrix}^T_* \frac{z_1 \ldots z_n}{(1-z_1)\ldots(1-z_n)},$$

where $\operatorname{adj}\gamma := |\det\gamma|\gamma^{-1}$ is the *adjugate* matrix of γ, while we recall that

$$\widetilde{r}([C]) = (-1)^n \left[\operatorname{adj}\begin{pmatrix} r_1 & \ldots & r_n \end{pmatrix}^{-1}\right]_* \frac{z_1 \ldots z_n}{(1-z_1)\ldots(1-z_n)},$$

and we compute that this expression in fact equal to the Shintani generating function $f_{C^\vee}(1)$ of the *dual* cone $C^\vee$, whose bounding rays are $r_1^\vee, \ldots, r_n^\vee$, defined by

$$\langle r_i^\vee, r_j \rangle = \begin{cases} \mathbb{R}_+, & i = j \\ 0, & i \neq j \end{cases}.$$

More intrinsically, the dual $C^\vee$ of *any* cone C can be defined as the locus pairing positively with C under $\langle -, - \rangle$; for simplicial cones, one can check this coincides with the definition in terms of bounding rays.

Remark 4.6. This "conical duality" between the two constructions is observed also in the following phenomenon: as we have seen previously, the realization of a wedge, as defined in Lemma 2.14, is generally nonzero under $\widetilde{r}$, while it is easy to compute that $f_C = 0$ for any wedge C, since

$$f_{\mathbb{R}}(z) = f_{\mathbb{R}_+}(z) + f_{\mathbb{R}_{\leqslant 0}}(z) = \frac{z}{1-z} + \frac{1}{1-z^{-1}} = 0,$$

cf. [SH]). Conversely, by construction, $\widetilde{r}$ ignores any kind of degenerate lower-dimensional simplex, while f_C makes sense and is generally nonzero for lower-dimensional cones C. Indeed, the roles played by "wedge" and "lower-dimensional cone" are swapped; they are precisely the conical duals to each other. For a concrete example, observe that

$$\mathbb{R}_+ \oplus \mathbb{R} \subset \mathbb{R}^2$$

has realization $z_1/(1-z_1)$, which is the same as the Shintani generating function for its conical dual $\mathbb{R}_+ \oplus \{0\} \subset \mathbb{R}$. Note also that under this duality, the fundamental class $[S^{n-1}]$ corresponds to the degenerate cone $\{0\}$; the realization $\widetilde{r}$, respectively the association of the Shintani generating function, send these to the constant function 1.

Remark 4.7. A consequence of the previous remark is in the *additivity* of $\widetilde{r}$, as opposed to the Shintani generating functions: when we have a decomposition of top-dimensional simplices/cones

$$[C_1] + [C_2] = [C_3],$$

[10]To be strict, here we should write $[\overline{C \cap S^{n-1}}]$ since C is a relatively open cone and we want a closed spherical chain; however, since the open/closed distinction does not exist within the chain complex, and the cone/simplex identification is very simple, for convenience we simply write $[C]$.

we have proven previously that $\widetilde{r}([C_1]) + \widetilde{r}([C_2]) = \widetilde{r}([C_3])$. However, it is not *quite* the case that

$$f_{C_1} + f_{C_2} = f_{C_3}.$$

Indeed, from the "naive" generating function definition (4.4), one sees that this *almost* is obvious from definition, except that the terms coming from lower-dimensional cones in the intersection $\overline{C_1} \cap \overline{C}_2$ will appear on the right-hand side, but not on the left-hand side: thus, this "obvious" additivity needs to be corrected by the intervention of lower-dimensional conical faces. This phenomenon poses a delicate problem in the construction of "true" Shintani domains; see discussion further below. Meanwhile, the additivity coming from $\widetilde{r}$ is less obvious on the level of expanding out infinite monomial sums (though it can be deduced by canceling out generating functions of wedges), but holds on the nose. In the terminology of [GP], the latter is "N-additivity," as opposed to the naive "M-additivity" needing correction from lower-dimensional faces. The duality between these types of additivity is well-known in convexity theory and linear programming; see, for example, [Bv, Chapter 1, Problem 3].

The above two remarks can be viewed more structurally in the following way: let $\mathcal{K}_{\mathbb{Q}}$ be the module of functions on $\mathbb{R}^n$ generated by indicator functions of relatively open rational simplicial cones, and let and $\mathcal{L}_{\mathbb{Q}}$ be those generated by indicator functions of (relatively open) wedges, as in [SH] and [CDG]. Notice then that the Shintani generating function can be viewed as an association[11]

$$f_\bullet : \mathcal{K}_{\mathbb{Q}}/\mathcal{L}_{\mathbb{Q}} \to \mathcal{M}_{\mathbb{G}_m^n}, 1_C \mapsto f_C$$

since the Shintani generating function of a wedge is zero; from this viewpoint, a Shintani decomposition $\mathscr{S}$ is simply the corresponding linear combination of indicator functions of its cones. This association satisfies the $\mathrm{GL}_n(\mathbb{Q})$-equivariance property

$$f_{\psi\circ\gamma^{-1}} = (\mathrm{adj}\, \gamma^T)_* f_\psi$$

for $\psi \in \mathcal{K}_{\mathbb{Q}}/\mathcal{L}_{\mathbb{Q}}$. We then have:

Proposition 4.8. *The association*

$$[\Delta] \mapsto 1_{(\mathbb{R}_+\Delta)^\vee},$$

on a positively-oriented simplex Δ, defines a map

$$\delta : \widetilde{C}(n) \to \mathcal{K}_{\mathbb{Q}}/\mathcal{L}_{\mathbb{Q}}, [\Delta] \mapsto 1_{(\mathbb{R}_+\Delta)^\vee}$$

satisfying the equivariance property

$$\delta(\gamma[\Delta]) = \delta([\Delta]) \circ \gamma^T$$

for $\gamma \in \mathrm{GL}_n(\mathbb{Q})$. Furthermore, $\widetilde{r}(\bullet) = f_{\delta(\bullet)}(1)$.

Proof. This is a direct consequence of the convex cone duality of [Bv, IV, Theorem 1.6], once one notes that the duals of lower-dimensional cones are precisely wedges (and identifies the "polar" as the negative of the dual). The GL_n-equivariance property is an immediate consequence of the definition of dual cone, and the asserted equality of "realization maps" is then a direct consequence of the formulas. ∎

Now, associated to the coordinatization α of $F \otimes \mathbb{R}$ is an embedding

$$\iota_\alpha : U \to \mathrm{GL}_n(\mathbb{Z}).$$

[11] This is just a restricted version of the Solomon-Hu pairing defined in [SH].

It was an observation of Colmez, generalized in [CDG] (and independently by Diaz y Diaz–Friedman), that the value of $\theta^{S^{n-1}}$ on a set of generators for U^+, when antisymmetrized, gives a fundamental Shintani domain "up to boundaries," i.e. up to the lower-dimensional conical faces. In particular, it follows from [CDG, Theorem 1.5]:

Proposition 4.9. *If $u_1, \ldots, u_{n-1}$ is set of generators of U^+, then the set of $(n-1)!$ simplices*

$$\Delta^\circ(v, u_{\tau(1)}v, \ldots, u_{\tau(1)} \ldots u_{\tau(n-1)}v),$$

for any $v \in I^+$, as τ ranges over S_{n-1}, contains a unique U^+-translate of any $x \in (F \otimes \mathbb{R})^+/\mathbb{R}_+^\times$ not in the orbit of their boundary faces. Equivalently, the associated open cone contains a unique U^+ translate of any $x \in (F \otimes \mathbb{R})^+$ not in a boundary orbit.

Fix any ray v in $(F \otimes \mathbb{R})^+ \overset{\alpha}{\subset} \mathbb{R}^n$, and choose an ordering of basis $u_1, \ldots, u_{n-1}$ of U^+ such that $\Delta^\circ(v, u_1 v, \ldots, u_1 \ldots u_{n-1} v)$ is positively oriented with respect to the standard orientation of $\mathbb{R}^n$ (given by the ordered standard basis $e_1, \ldots, e_n$). Note that $\det(v, u_1 v, \ldots, u_1 \ldots u_{n-1} v)$ is nowhere vanishing in $(F \otimes \mathbb{R})^+$; thus, by continuity, we may define:

Definition 4.10. *We define a choice of fundamental class $c_{U^+}^\alpha \in H_{n-1}(U^+, \mathbb{Z})$ corresponding to the orientation such that $\Delta^\circ(v, u_1 v, \ldots, u_1 \ldots u_{n-1} v)$ is positively oriented for* any *$v \in (F \otimes \mathbb{R})^+$. (Note that the dependence on α is only up to sign.)*

Remark 4.11. More generally, there are 2^n orthants in $F \otimes \mathbb{R}$ corresponding to possible combinations of signs at real places of F; their disjoint union is the complement of the zero locus of $v \mapsto \det(v, u_1 v, \ldots, u_1 \ldots u_{n-1} v)$, so the orientation of $\Delta^\circ(v, u_1 v, \ldots, u_1 \ldots u_{n-1} v)$ is constant on each of them. This will be $+1$ on all the orthants with an even number of negative signs, and -1 otherwise: this is clear if one simultaneously diagonalizes $u_1, \ldots, u_{n-1}$, and notices that the matrices $\mathrm{diag}(\pm 1, \ldots, \pm 1)$ (which is orientation-preserving exactly when the number of -1s is even) in the resulting eigenbasis transitively permute the simplices along with their corresponding orthants.

Now take a cocycle like $\theta^{S^{n-1}}(n)|_{U^+}$, but lifting to some general $\Delta(v) \in S^{n-1}(\mathbb{Q})$ instead of $[\Delta(e_1)]$ from $1 \in \mathrm{Chains}(n)_0$, representing the class $\Theta^{S^{n-1}}(n)|_{U^+} \in H^{n-1}(U^+, C(n))$. Then the cap product

$$\theta^{S^{n-1}}(n) \frown c_{U^+}^\alpha = \sum_{\sigma \in S_{n-1}} (-1)^{\mathrm{sgn}\,\sigma} [\Delta(v, u_{\tau(1)}v, \ldots, u_{\tau(1)} \ldots u_{\tau(n-1)}v)] \in H_0(U^+, C(n)) \tag{4.5}$$

is a formal sum of simplices whose $\mathbb{R}^+$-span is a Shintani domain, up to lower-dimensional cones.

Previous cohomological approaches, such as [Hill] or [CDG], used auxiliary data (a lexicographic order, respectively a "Colmez" perturbation vector) to find the lower-dimensional cones which correct the top-dimensional term corresponding to the "fake Shintani domain" $f_C(\xi)$, for

$$C = \mathbb{R}_+ \Delta^\circ(v, u_{\tau(1)}v, \ldots, u_{\tau(n-1)} \ldots u_{\tau(1)}v),$$

and thus obtain totally real L-values; for example, [CDG, Theorem 2.1] shows that

$$D = (\text{linear combination of indicators of lower dim. cones}) + \sum_{\tau \in S_{n-1}} 1(\mathbb{R}_+ \Delta^\circ(v, u_{\tau(1)}v, \ldots, u_{\tau(n-1)} \ldots u_{\tau(1)}v) \in \mathcal{K}_{\mathbb{R}^n}$$

is a "signed Shintani domain," which is to say formally satisfies the requirements of Theorem 4.4 if we view $f_\bullet$ as a function on $\mathcal{K}_\mathbb{Q}/\mathcal{L}_\mathbb{Q}$. However, *our* cocycle has values which look like

$$\widetilde{r}([\Delta(v, u_{\tau(1)}v, \ldots, u_{\tau(n-1)} \ldots u_{\tau(1)}v)]),$$

which as we have seen, are actually *not* the generating functions considered in those articles. Thus, even ignoring the issue of lower-dimensional faces, it is not immediately clear how Shintani's result Theorem 4.4 can be used to relate our cocycle to L-values of totally real fields; we are a "conical duality" away.

The solution lies in the following observation: the target $\mathcal{K}_{\mathbb{Q}}/\mathcal{L}_{\mathbb{Q}}$ of the duality morphism δ is very similar in nature to $\widetilde{\mathrm{C}}(n)$; as remarked previously, the difference is that the latter "cares about" wedges, while the former "cares about" lower-dimensional cones. If we could define the realization map $\widetilde{r}$ on the *former* module instead of the latter, then $\widetilde{r} \circ \delta$ *would* carry the class of our "fake fundamental domain" to the "main" (top-dimensional) term contributing to a Shintani function for a totally real field F. But $\widetilde{r}$ *does* care about wedges, and ignores lower-dimensional cones, so this strategy does not quite make sense, as written.

The work-around is that if we introduce certain stabilizations of our symbol complexes by torsion sections, then we can remove all the above obstructions: the classes of wedges and lower-dimensional cones alike are annihilated, so that δ becomes a kind of auto-duality, whereupon a realization extending $\widetilde{r}$ makes sense on the target. This stabilization will also remove the contribution of all lower-dimensional terms in associated Shintani generating functions, effectively making our fake fundamental domain as good as a real one. The resulting stabilized cocycle can then be directly related to a (stabilized) Shintani function for a totally real field F, and therefore to L-values.

4.3. **Stabilized duality and p-adic L-functions.** We introduce stabilizations at auxiliary integers, necessary to realize the previously-stated strategy for constructing and proving interpolation properties for our p-adic L-functions. Because δ is a duality map, there will be two dual notions of stabilization, which we will call *avoiding* and *smoothing*. Broadly speaking, "avoiding" corresponds to ensuring the poles of the values of $\Theta^{dR}(n)$ avoid zero, while "smoothing" enforces a degree-zero condition killing off lower-dimensional conical faces.

Remark 4.12. The Euler factors that appear corresponding to these two types of stabilization are precisely those corresponding to the two kinds of sets of places often labelled S and T in literature on Stark-type conjectures; e.g., in [DK].

4.3.1. *c-avoiding.* Let $c > 1$ be an integer prime to p; though it is not necessary, for later convenience we may as well take it to be prime. We first describe a "c-avoiding" modification of our symbol complex, as well as the resulting stabilization of $\Theta^{dR}(n)$; this latter class has the upside is that it is valued in functions holomorphic in a p-adic neighborhood of the identity.

Write

$$ {}_c\widetilde{\mathrm{Chains}}'(n)_i = \bigoplus_{\lambda \in \mathbb{P}^{n-1}(\mathbb{Z}/c)} {}_\lambda\widetilde{\mathrm{Chains}}(n)_i $$

where the definition of ${}_\lambda\widetilde{\mathrm{Chains}}(n)_\bullet[-1]$ needs to be explained: roughly, it is the augmented simplicial chain complex for the ind-triangulation of S^{n-1} consisting of simplices whose $< (n-1)$-dimensional faces do not contain any line reducing to λ modulo c. We will call a triangulation of S^{n-1} satisfying this property "λ-avoiding".

The *existence* of λ-avoiding triangulations is clear from the density of points reducing to any given ray in $(\mathbb{Z}/c)^n$ in any open set of S^{n-1}. The existence of a direct limit over *all* such triangulations (analogous to (2.2)) is slightly less obvious, but follows from just a slight refinement in the proof of Proposition 2.3: we need two λ-avoiding triangulations T_1 and T_2 to have a λ-avoiding common refinement. In that argument, when superimposing two λ-avoiding triangulations, all the resulting polyhedral faces of dimension $< (n-1)$ will automatically already avoid λ modulo c by assumption. To ensure all faces avoid λ, we pick the point of barycentric subdivision such that all the resulting new faces also avoid λ, rather than picking an arbitrary rational point, which again is clearly possible by density.

Thus, as in (2.2), we can define ${}_\lambda\widetilde{\mathrm{Chains}}(n)_\bullet[-1]$ as the direct limit over *all* chain complexes of x-avoiding triangulations, as before; it is naturally a subcomplex of $\widetilde{\mathrm{Chains}}(n)_\bullet$, and the inclusion is a quasi-isomorphism.

Clearly, pushforward by $\gamma \in \mathrm{GL}_n(\mathbb{Z}_{(c)})$ carries λ-avoiding triangulations to $\gamma\lambda$-avoiding triangulations, for the natural left action on lines. Then the complex ${}_c\widetilde{\mathrm{Chains}}'(n)_i$ carries a natural $\mathrm{GL}_n(\mathbb{Z})$ action, which permutes the summands

$$ {}_\lambda\widetilde{\mathrm{Chains}}(n)_i \to {}_{\gamma\lambda}\widetilde{\mathrm{Chains}}(n)_i $$

given simply by the pushforward $\gamma_* : S^{n-1} \to S^{n-1}$ on the level of chains. As representations of this group, we can identify

$$ {}_c\widetilde{\mathrm{Chains}}'(n)_0 \cong \mathbb{Z}\{\mathbb{P}^{n-1}(\mathbb{Z}/c)\}. $$

Similarly,

$$ H_n({}_c\widetilde{\mathrm{Chains}}(n)_\bullet) \cong \mathbb{Z}\{\mathbb{P}^{n-1}(\mathbb{Z}/c)\}(\mathrm{sgn}) $$

with the same action twisted by the sign character, since there is one fundamental class of the sphere for $\lambda \in \mathbb{P}^{n-1}(\mathbb{Z}/c)$. If we write $H^0 \subset \mathbb{Z}\{\mathbb{P}^{n-1}(\mathbb{Z}/c)\}$ for the submodule of degree-zero elements in $\mathbb{Z}\{\mathbb{P}^{n-1}(\mathbb{Z}/c)\}(\mathrm{sgn})$, we can define

$$ {}_c\widetilde{\mathrm{Chains}}(n)_\bullet := {}_c\widetilde{\mathrm{Chains}}'(n)_\bullet / H^0 \tag{4.6} $$

$$ {}_c\mathrm{Chains}(n)_\bullet := {}_c\widetilde{\mathrm{Chains}}'(n)_\bullet / H_n({}_c\widetilde{\mathrm{Chains}}(n)_\bullet), \tag{4.7} $$

and the latter complex is exact. The same argument in the proof of Theorem 3.1 (but skipping the motivic realization to pass directly to the $d\log$ regulator) shows that there is a $\mathrm{GL}_n(\mathbb{Z})$-equivariant realization map

$$ {}_c\tilde{r} : {}_c\widetilde{\mathrm{Chains}}(n)_n \to \mathcal{M}_{\mathbb{G}_m^n}(-\det)^{(0)} $$

given on acyclic simplicial generators by

$$ [\Delta(r_1, \ldots, r_n)]_\lambda \mapsto \sum_{x \in \lambda - \{0\}} \frac{1}{\det M} t_x^* M_* \frac{(-1)^n z_1 \ldots z_n}{(1 - z_1) \ldots (1 - z_n)} \tag{4.8} $$

where we here use the identification $\mu_c^n = \mathbb{G}_m^n[c] \cong (\mathbb{Z}/c)^n$ coming from considering $\mathbb{Z}^n$ as the homology lattice of $\mathbb{G}_m^n$, as we have been doing, to consider the various x as torsion points in $\mathbb{G}_m^n$, and

$$ M = \begin{pmatrix} r_1 & \ldots & r_n \end{pmatrix}. $$

(Here, one has to be careful that the (0) condition needs also to exclude the trace $[c]_*$; this applies going forward.) This realization quotients down to a map

$$ {}_c r : {}_c\mathrm{Chains}(n)_n \to \mathcal{M}_{\mathbb{G}_m^n}(\mathrm{sgn})^{(0)}/\mathrm{constants}. $$

By the λ-avoiding condition, the image of ${}_c r$ lands in the intersection of $A_p^{(n)}$ for *all* p prime to c. Fixing some such p, we obtain by lifting the fixed class

$$ \sum_{\lambda \in \mathbb{P}^{n-1}(\mathbb{Z}/c)} \lambda \in {}_c\widetilde{\mathrm{Chains}}(n)_0 $$

a cocycle

$$ {}_c\Theta^{dR}(n) \in H^{n-1}(\mathrm{GL}_n(\mathbb{Z}), A_p^{(n)}(\mathrm{sgn})/\mathbb{Z}_p) $$

which, via Cartier duality (as discussed previously), yields a cocycle

$$ {}_c\Theta^{dR}_{\mathbf{D}_p}(n) \in H^{n-1}(\mathrm{GL}_n(\mathbb{Z}), \mathbf{D}^{(0)}((\mathbb{Z}_p^n)^\vee, \mathbb{Z}_p)(\mathrm{sgn})/\langle \delta_0 \rangle) $$

where δ_0 is the atomic ("Dirac delta") measure at zero.

Remark 4.13. There is a simple relation between ${}_c\Theta^{dR}(n)$, viewed as valued in $\mathcal{M}_{\mathbb{G}_m^n}(\mathrm{sgn})^{(0)}$, and $\Theta^{dR}(n)$: we can view both "spherical chain complexes" $\mathrm{GL}_n(\mathbb{Z})$-equivariantly inside $\bigoplus_{x\in\mu_c^n}\mathrm{Chains}(n)_\bullet$, by including ${}_c\mathrm{Chains}(n)$ by embedding its λ-summand into all the nonzero x contained in λ, and inclusion at the identity section for $\mathrm{Chains}(n)$. The class $\Theta^{dR}(n)$ is obtained by lifting $\{1\}\in\mathrm{Chains}(n)_0$, while ${}_c\Theta^{dR}(n)$ is obtained by lifting $([c]^*-1)\{1\}$; then by functoriality under $[c]^*$ of the lifting process in group cohomology, we find

$$(c^n[c]^*-1)\Theta^{dR}(n)={}_c\Theta^{dR}(n)$$

after noting that ${}_cr\circ[c]^*=c^n[c]^*\circ{}_cr$.[12]

4.3.2. *Restriction to a nonsplit torus.* We now restrict to a nonsplit torus corresponding to a totally real field F of degree n: as before, we pick a fractional ideal I of F, but instead of identifying it with $\mathbb{Z}^n$ as in the original Shintani method, we will fix instead an isomorphism

$$\alpha: I\xrightarrow{\sim}(\mathbb{Z}^n)^\vee$$

to the *dual*, which is equivariant for $U\subset\mathrm{GL}_n(\mathbb{Z})$ for the usual unit multiplication on the source and the standard dual representation on the target. The definition of the orientation $c^\alpha_{U^+}\in H^{n-1}(U^+,\mathbb{Z})$ is as in Definition 4.10, with respect to the standard orientation on $(\mathbb{R}^n)^\vee$; i.e., corresponding to the ordering $e_1^\vee,\ldots,e_n^\vee$.

We write $\iota_\alpha: U\hookrightarrow\mathrm{GL}_n(\mathbb{Z})$ for the resulting inclusion; remark that any scaling $\alpha\circ[\times a]$ for $a\in F$ is U-equivariant for the same inclusion ι_α. Conversely, ι_α can be recovered from the data of α by writing out the matrix action of U on I, then taking the inverse transpose, so each ι_α is uniquely associated to an "isogeny class" of coordinatizations α.

Then α induces a U-equivariant identification

$$\mathbf{D}^{(0)}((\mathbb{Q}_p^n)^\vee,\mathbb{Z}_p)\xrightarrow{\sim}\mathbf{D}^{(0)}(F_p,\mathbb{Z}_p),\mu\mapsto(\varphi\mapsto\mu(\varphi\circ\alpha^{-1}))\tag{4.9}$$

via which we view ${}_c\Theta^{dR}_{\mathbf{D}_p}(n)|_{U^+}$ as living inside $H^{n-1}(U^+,\mathbf{D}(F_p,\mathbb{Z}_p)^{(0)}/\langle\delta_0\rangle)$.

4.3.3. *Stabilizing at ideals of F.* We record some transformations of our cocycle, and properties thereof, which only make sense *after* restriction via some $\iota_\alpha: U\hookrightarrow\mathrm{GL}_n(\mathbb{Z})$ associated to a dual coordinatization α as in the previous section. Via considering the rationalization of α, we see that ι_α extends uniquely to an algebra embedding $F\to M_2(\mathbb{Q})$ which we continue to denote ι_α. Then for any integral ideal $\mathfrak{a}\subset\mathcal{O}_F$, it makes sense to speak of the $\mathfrak{a}$-torsion $\mathbb{G}_m^n[\mathfrak{a}]$, as the torsion points annihilated by the action of all elements $\mathfrak{a}$ via ι_α; this torsion group has size $\mathbf{N}\mathfrak{a}$. Notice that I is here identified with the character lattice of $\mathbb{G}_m^n$, so that $\mathbb{G}_m^n[\mathfrak{a}]=\hom_{\mathcal{O}_F}(I,\mathcal{O}_F/\mathfrak{a})=I^\vee\otimes_{\mathcal{O}_F}\mathcal{O}_F/\mathfrak{a}$ for example. The natural multiplication action of $\mathcal{O}_F$ on I can be thought of a *pullback* action in this optic.

This has the following upshot: if we *first* restrict to the image of $F^\times\xrightarrow{\iota_\alpha}\mathrm{GL}_n(\mathbb{Q})$, then we may define the $F^\times$-equivariant complex ${}_\mathfrak{c}\mathrm{Chains}(\alpha)_\bullet$ for any prime ideal $\mathfrak{c}$ of F (which we choose prime to p):

Definition 4.14. *The U-module ${}_\mathfrak{c}\mathrm{Chains}(\alpha)_\bullet$ is defined similarly to ${}_c\mathrm{Chains}(n)_\bullet$, except the direct sum is taken over only the lines $\lambda\in\mathbb{P}^{n-1}(\mathbb{Z}/c)$ contained in the subspace of $(\mathbb{Z}/c)^n$ corresponding to $\mathbb{G}_m^n[\mathfrak{c}]$ (for the action of $F^\times$ on $\mathbb{G}_m$ given by ι_α).*

The subspace of $(\mathbb{Z}/c)^n$ indicated can also be described as the dual of the kernel of the natural projection $I/c\to I/\mathfrak{c}$, recalling that I is identified with $(\mathbb{Z}^n)^\vee$.

[12]The extra scalar c^n comes from the det-twist in the contraction from n-forms to functions: in general, we have $\gamma^*(f\,dt_1\wedge\ldots\wedge dt_n)=(\det\gamma)(\gamma^*f)\,dt_1\wedge\ldots\wedge dt_n$.

Then the corresponding cycle $\mathbb{G}_m^n[\mathfrak{c}] - \{1\} \in {}_{\mathfrak{c}}\mathrm{Chains}(\alpha)_\bullet$ is $\iota_\alpha(U)$-fixed, and we have a realization map

$$ {}_{\mathfrak{c}}r : {}_{\mathfrak{c}}\mathrm{Chains}(\alpha)_n \to \mathcal{M}_{\mathbb{G}_m^n}(\mathrm{sgn})^{(0)}/\mathrm{constants}. $$

defined exactly analogously to ${}_c r$ via a sum of translates (4.8). We therefore can define de Rham classes

$$ {}_{\mathfrak{c}}\Theta^{dR}(\iota_\alpha) \in H^{n-1}(U, A_p^{(n)}/\mathrm{constants}) $$

as in the case of c-stabilization, but only after restricting to U along ι_α.

Since U^+ preserves the angular domain $(F \otimes \mathbb{R})^+ \subset \mathbb{R}^n$ (along with all other orthants coming from combinations of signs at places of F, all lifted chains in any Δ-extension always are contained in such an orthant and never contain any full line. Thus, the Euler cocycle $\varepsilon_n|_{U^+}$ is *identically* zero regardless of Δ-extension chosen, and we can canonically lift ${}_{\mathfrak{c}}\Theta^{dR}(\iota_\alpha)$ over the orientation obstruction, allowing us to view these classes (following the convention of Section 4.3.3)

$$ {}_{\mathfrak{c}}\Theta^{dR}(\iota_\alpha) \in H^{n-1}(U, A_p^{(n)}), {}_{\mathfrak{c}}\Theta^{dR}_{\mathbf{D}_p}(\alpha) = (\alpha^{-1})_*\kappa_*{}_{\mathfrak{c}}\Theta^{dR}(\alpha) \in H^{n-1}(U, \mathbf{D}(F_p, \mathbb{Z}_p)^{(0)}) \tag{4.10} $$

without quotienting by the images of the orientation obstruction. Another way to view this Euler obstruction vanishing is that we have exact U-equivariant subcomplexes

$$ {}_{\mathfrak{c}}\mathrm{Chains}(\alpha)_\bullet \subset {}_{\mathfrak{c}}\mathrm{Chains}(\alpha)_\bullet $$

consisting of chains contained entirely in the (contractible) totally positive orthant of the sphere, and we define the above classes by lifting in these complexes.

If $\mathfrak{c}$ is principal and generated by an element of positive norm, say $\beta \in \mathcal{O}_F^+$, then we also have $((\mathbf{N}\mathfrak{c})[\beta]^* - 1)\Theta^{dR}(n)|_U = {}_{\mathfrak{c}}\Theta^{dR}(\iota_\alpha)$, if one uses the totally positive orthant to lift the restriction of $\Theta^{dR}(n)$ over the Euler class in the same way, where here we use that the determinant of the action of β is its norm as an element of F.

4.3.4. *p-adic L-functions.* We now define ${}_{\mathfrak{c}}\zeta_p^{I,\alpha}$ to be the image of ${}_{\mathfrak{c}}\Theta^{dR}_{\mathbf{D}_p}(\alpha)$ under the composition

$$ H^{n-1}(U^+, \mathbf{D}(F_p, \mathbb{Z}_p)^{(0)}) \xrightarrow{\frown c_{U^+}^\alpha} H_0(U^+, \mathbf{D}(F_p, \mathbb{Z}_p)^{(0)})) \to \mathbf{D}(F_p/U^+, \mathbb{Z}_p)^{(0)} \tag{4.11} $$

Here, the last map is the natural map

$$ H_0(U^+, \mathbf{D}(F_p, \mathbb{Z}_p)^{(0)}) \to H_0(U^+, \mathbf{D}(F_p, \mathbb{Z}_p))^{(0)} \twoheadrightarrow \mathbf{D}(F_p/U^+, \mathbb{Z}_p)^{(0)}. $$

Explicitly, if φ is a U-invariant compactly supported continuous function on F_p, which we view as a function on $(\mathbb{Q}_p^n)^\vee$ via

$$ (\mathbb{Q}_p^n)^\vee \xrightarrow{\alpha^{-1}} F_p \xrightarrow{\varphi} \mathbb{Q}_p, $$

we have

$$ \int_{F_p/U^+} \varphi(s)\, d{}_{\mathfrak{c}}\zeta_p^{I,\alpha}(s) = \int_{(\mathbb{Q}_p^n)^\vee} \varphi(\alpha^{-1}(t))\, d({}_{\mathfrak{c}}\Theta^{dR}_{\mathbf{D}_p}(n)|_{U^+} \frown c_{U^+}^\alpha). \tag{4.12} $$

For $\mathfrak{c} = (c)$ an ideal of $\mathbb{Z}$, these zeta elements are specializations of the "global" cocycle ${}_c\Theta^{dR}(n)$ (up to the orientation ambiguity); otherwise, it is only a specialization of ${}_{\mathfrak{c}}\Theta^{dR}(\iota_\alpha)$.

Recall now that the coordinatization α determines ι_α, but not vice versa; scaling α by elements of F (and therefore identifying the lattice $(\mathbb{Z}^n)^\vee$ with different ideals in the same class) does not change the latter. We consider the dependence of ${}_c\Theta^{dR}(\alpha)$, and ${}_{\mathfrak{c}}\zeta_p^{I,\alpha}$ on α (instead of merely ι_α).

In particular, if we start with some fixed identification $\alpha : I \to (\mathbb{Z}^n)^\vee$, we can consider also the scaling

$$\alpha \circ [t^{-1}] : (t)I \to (\mathbb{Z}^n)^\vee$$

for any $t \in F^\times$; as noted, this is equivariant for the same $\iota_\alpha : U \hookrightarrow \mathrm{GL}_n(\mathbb{Z})$. As with $c^\alpha_{U^+}$, the dependence of the p-adic L-function on α is quite weak; rescaling by t results in the same information up to a flip of sign:

Proposition 4.15. *We have the relation*

$$(\mathrm{sgn}\,\mathbf{N}t)[t]_{*c}\zeta_p^{I,\alpha} = {}_c\zeta_p^{(t)I,\alpha\circ[t^{-1}]}. \tag{4.13}$$

for the map

$$[t]_* : \mathbf{D}(F_p/U^+, \mathbb{Z}_p)^{(0)} \to \mathbf{D}(F_p/U^+, \mathbb{Z}_p)^{(0)}, \mu \mapsto (\varphi \mapsto \mu(\varphi \circ [t])).$$

Proof. Starting from the fact that ${}_{\mathfrak{c}}\Theta^{dR}(\iota_\alpha)$ only depends on ι_α, we obtain immediately that

$$[t]_{*\mathfrak{c}}\Theta^{dR}_{\mathbf{D}_p}(\alpha) = {}_{\mathfrak{c}}\Theta^{dR}_{\mathbf{D}_p}(\alpha \circ [t^{-1}]).$$

Then from the functoriality under $[t]_*$ of the sequence of maps (4.11), this amounts to showing that $c^{\alpha\circ[t^{-1}]}_{U^+} = (\mathrm{sgn}\,\mathbf{N}t)c^\alpha_{U^+}$, which follows from Remark 4.11. ☐

This provides a strong sense in which the information in our p-adic L-functions only depends on the narrow ideal class of I, and in fact only on the wide ideal class "up to sign". This will be used later in Section 4.4.3 to build p-adic L-functions for ray class characters, and prove 2-adic congruences.

We have thus defined certain "p-adic L-function" elements, but have not justified the name; we need an interpolation theorem:

Theorem 4.16. *Let $\psi : I^+/U^+ \to \mathbb{Z}_p^\times$ be any locally constant function and $k \geqslant 0$ any integer. Then we have the specialization property*

$$(-1)^{n-1}\int_{F_p/U^+} \psi(t)\mathbf{N}(t)^k \, d_{\mathfrak{c}}\zeta_p^{I,\alpha}(t) = \zeta_I(\psi, -k) - \mathbf{N}\mathfrak{c}^k\zeta_{\mathfrak{c}I}(\psi, -k).$$

Note that Theorem 1.3 from the introduction is an immediate corollary. Most of the remainder of this article will be dedicated to proving this interpolation theorem, by finding a duality reconciling the (stabilized) values of $\Theta^{dR}(n)$ to the (stabilized) classical Shintani method for F.

4.3.5. *c-smoothing.* We now describe the "c-smoothed" modification of $\widetilde{\mathrm{Chains}}(n)$: in fact, this modification will set all wedge classes to zero, so it is actually rather a "Orlik-Solomon"-flavored complex (as in Section 2.3.3), moreso than a "spherical chains" flavored one. As with c-avoiding, after describing the case of smoothing at an integer resulting in a $\mathrm{GL}_n(\mathbb{Z})$-cocycle, we will also indicate a generalization to smoothing at an arbitrary ideal $\mathfrak{c} \subset F$; this yields only a U-cocycle when $\mathfrak{c}$ is not a rational ideal. Again, we emphasize smoothing at integers more for expository purposes, as working with GL_n-actions makes the underlying linear algebraic structure conceptually clearer.

We write $\check{\mathbb{P}}$ to mean the dual projective space (parameterizing hyperplanes) and $\check{G}$ for the Pontryagin dual of a group G. Then we define the c-smoothed complex

$${}^c\mathrm{OS}(n)_i = \bigoplus_{\eta \in \check{\mathbb{P}}^{n-1}(\mathbb{Z}/c)} {}^\xi\mathrm{OS}(n)_i$$

where ${}^\eta\mathrm{OS}(n)_\bullet$ is defined as the Orlik-Solomon complex for the lines $\ell \in \mathbb{P}^{n-1}(\mathbb{Q}) = \mathbb{P}^{n-1}(\mathbb{Z})$ not contained in η modulo c. Each summand is exact as a complex by the usual Orlik-Solomon formalism, so the whole complex is as well. In top degree, we will also $\mathrm{St}(n)$ instead of $\mathrm{OS}(n)_n$, continuing earlier "Steinberg representation" notation. We define the $\mathrm{GL}_n(\mathbb{Z})$-action on this complex by

$$^{\eta}\mathrm{OS}(n)_i \to {}^{\gamma\eta}\mathrm{OS}(n)_i, ([\ell_1] \wedge \ldots \wedge [\ell_i])_\eta \mapsto [([\gamma\ell_1] \wedge \ldots \wedge [(\gamma\ell_i])_{\gamma\eta}.$$

Proposition 4.17. *There is a* $\mathrm{GL}_n(\mathbb{Z})$*-equivariant realization map*

$$^{c}r : {}^{c}\mathrm{St}(n) \to (\mathcal{M}_{\mathbb{G}_m^n}(\mathrm{sgn}))^{(0)}$$

given by

$$([\ell_1] \wedge \ldots \wedge [\ell_n])_\eta \mapsto \sum_{\substack{\xi \in \widetilde{\mu_c}^n \\ \ker \xi = \eta}} \sum_{x \in \mu_c^n} \xi(x) t_x^* (\det M)^{-1} M_* \frac{(-1)^n z_1 \ldots z_n}{(1 - z_1) \ldots (1 - z_n)} \tag{4.14}$$

where $M = \begin{pmatrix} \ell_1 & \ldots & \ell_n \end{pmatrix}$*; here, the notation as usual means we pick any integral generator of the line (or in fact, any integral point, by the usual trace-fixedness argument).*

Proof. We saw previously, in Section 2.3.3, that the Orlik-Solomon relations are generated by the spherical chain relations together with the "wedge" relations, where the latter amount to checking that the formula (4.14) is independent of replacing a column in M with its negative: our previous trace-fixedness arguments show that the formula is independent of scaling the generators by $[a]$ for $a \in \mathbb{N}$, but not for scaling by $[-1]$.

The proof that the stellar subdivision relations hold in the image of ^{c}r is identical to the argument for Theorem 1.1, so it suffices to check latter wedge relations. Indeed, if we identify $\ell_1, \ldots, \ell_n$ with a set of integral generators, let

$$M' := \begin{pmatrix} -\ell_1 & \ldots & \ell_n \end{pmatrix}$$

be the matrix for the same set of integral generators, with one sign flipped. Then we compute that the difference between the expression (4.14) for M and M' is

$$\sum_{\substack{\xi \in \widetilde{\mu_c}^n \\ \ker \xi = \eta}} \sum_{x \in \mu_c^n} \xi(x) t_x^* (\det M)^{-1} M_* \left(\frac{(-1)^n z_1 \ldots z_n}{(1 - z_1) \ldots (1 - z_n)} + \frac{(-1)^n z_1^{-1} \ldots z_n}{(1 - z_1^{-1}) \ldots (1 - z_n)} \right)$$

$$= (\det M)^{-1} \sum_{\substack{\xi \in \widetilde{\mu_c}^n \\ \ker \xi = \eta}} \sum_{x \in \mu_c^n} \xi(x) t_x^* M_* \frac{(-1)^{n-1} z_2 \ldots z_n}{(1 - z_2) \ldots (1 - z_n)}.$$

If we split this sum over x into cosets for the order-c subgroup $(\ell_1)_* \mu_c \subset \mu_c^n$, then one computes that

$$t_x^* M_* \frac{(-1)^{n-1} z_2 \ldots z_n}{(1 - z_2) \ldots (1 - z_n)}$$

is constant on each coset, and ξ has sum zero over each coset since it is a nontrivial character on $(\ell_1)_* \mu_c$. The result follows. $\square$

Inside $^{c}\mathrm{OS}(n)_0 \cong \mathbb{Z}\{\check{\mathbb{P}}^{n-1}(\mathbb{Z}/c)\}$, we have the $\mathrm{GL}_n(\mathbb{Z})$-fixed element

$$- \sum_{\eta \in \check{\mathbb{P}}^{n-1}(\mathbb{Z}/c)} \eta,$$

which, if viewed as a function on μ_c^n, via considering η as the sum over all characters with kernel η and then as a formal linear combination of elements thereof in the usual way where the values of the functions become the coefficients, corresponds to the torsion cycle $([c]^* - c^n)\{1\}$.

As usual, lifting this element and taking the realization ${}^c r$ then affords us a cocycle

$$ {}^c\Theta^{dR}(n) \in H^{n-1}(\mathrm{GL}_n(\mathbb{Z}), \mathcal{M}_{\mathbb{G}_m^n}(\mathrm{sgn})^{(0)}) $$

which, by the same argument used in Remark 4.13, satisfies $(c^n[c]^* - c^n)\Theta^{dR}(n) = {}^c\Theta^{dR}(n)$.

Now if we first restrict to $U \xrightarrow{\iota_\alpha} \mathrm{GL}_n(\mathbb{Z})$, we can likewise define a $\mathfrak{c}$-smoothed complex with U-action

$$ {}^{\mathfrak{c}}\mathrm{OS}(\alpha)_i = {}^{\eta_{\mathfrak{c}}}\mathrm{OS}(\alpha)_i $$

where η is the hyperplane which is the kernel of projecting $\mathbb{G}_m[c] \to \mathbb{G}_m[\mathfrak{c}]$, and an $\iota_\alpha(U)$-equivariant realization map ${}^{\mathfrak{c}}r$: ${}^{\mathfrak{c}}\mathrm{St}(\alpha) \to (\mathcal{M}_{\mathbb{G}_m^n}(\mathrm{sgn}))^{(0)}$, where as usual the Steinberg notation means the top degree module in the Orlik–Solomon complex. Then the $\iota_\alpha(U)$-fixed element

$$ \sum_{\xi \in \widetilde{\mathbb{G}_m^n[\mathfrak{c}]} - \{1\}} -\xi $$

(viewing these characters as being inflated via the natural projection $\mathbb{G}_m^n[c] \to \mathbb{G}_m^n[\mathfrak{c}]$) affords us a cocycle

$$ {}^{\mathfrak{c}}\Theta^{dR}(\alpha) \in H^{n-1}(U, (\mathcal{M}_{\mathbb{G}_m^n}(\mathrm{sgn}))^{(0)}). $$

Again, if $\mathfrak{c} = (\beta)$ generated by a positive-norm element, we have $\mathbf{N}\mathfrak{c}([\beta]^* - 1)\Theta^{dR}(n)|_U = {}^{\mathfrak{c}}\Theta^{dR}(\alpha)$, generalizing the case of $\mathfrak{c} = (c)$.

4.3.6. *Stabilized duality.* We now let b and c be distinct rational primes, which are also distinct from p. With identical arguments, we can combine the stabilizations of the preceding two sections to be simultaneously b-avoiding and c-smooth, resulting in an exact homological complex

$$ {}^c_b\mathrm{OS}(n)_\bullet = \bigoplus_{\eta \in \check{\mathbb{P}}^{n-1}(\mathbb{Z}/c)} \bigoplus_{\lambda \in \mathbb{P}^{n-1}(\mathbb{Z}/c)} {}^\eta_\lambda\mathrm{OS}(n)_\bullet $$

where the summands consisting of symbols whose $\leqslant (n-1)$-dimensional spans avoid λ, and whose lines modulo d are not contained in η. The $\mathrm{GL}_n(\mathbb{Z})$-invariant class

$$ - \sum_{\lambda \in \mathbb{P}^{n-1}(\mathbb{Z}/c)} \sum_{\eta \in \check{\mathbb{P}}^{n-1}(\mathbb{Z}/c)} 1_{\eta,\lambda} $$

yields a class ${}^c_b\Theta^{St}(n) \in H^{n-1}(\mathrm{GL}_n(\mathbb{Z}), {}^c_b\mathrm{St}(n))$. The $\mathrm{GL}_n(\mathbb{Z})$-equivariant realization map

$$ {}^c_b r : {}^c_b\mathrm{St}(n) \to A_p(\mathrm{sgn})^{(n)} $$

results in

$$ {}^c_b\Theta^{dR}(n) := {}^c_b r_* {}^c_b\Theta^{St}(n) \in H^{n-1}(\mathrm{GL}_n(\mathbb{Z}), A_p(\mathrm{sgn})^{(n)}) $$

such that

$$ (c^n[c]^* - c^n)(b^n[b]^* - 1)\Theta^{dR}(n) = (c^n[c]^* - c^n)_b\Theta^{dR}(n) = (b^n[b]^* - 1)^c\Theta^{dR}(n) = {}^c_b\Theta^{dR}(n). $$

Analogously, for $\mathfrak{b}$ and $\mathfrak{c}$ distinct primes of F not dividing p, we have a doubly stabilized complex ${}^{\mathfrak{c}}_{\mathfrak{b}}\mathrm{OS}(\alpha)_\bullet$, with top-degree module ${}^{\mathfrak{c}}_{\mathfrak{b}}\mathrm{St}(\alpha)$ and $\iota_\alpha(U)$-equivariant realization map ${}^{\mathfrak{c}}_{\mathfrak{b}}r : {}^{\mathfrak{c}}_{\mathfrak{b}}\mathrm{St}(\alpha) \to A_p(\mathrm{sgn})^{(n)}$, resulting in classes

$$ {}^{\mathfrak{c}}_{\mathfrak{b}}\Theta^{St}(\alpha) \in H^{n-1}(U, {}^{\mathfrak{c}}_{\mathfrak{b}}\mathrm{St}(\alpha)); \quad {}^{\mathfrak{c}}_{\mathfrak{b}}\Theta^{dR}(\alpha) = {}^{\mathfrak{c}}_{\mathfrak{b}}r_* {}^{\mathfrak{c}}_{\mathfrak{b}}\Theta^{St}(\alpha) \in H^{n-1}(U, A_p(\mathrm{sgn})^{(n)}) $$

such that

$$ (\mathbf{N}\mathfrak{c}[\epsilon]^* - \mathbf{N}\mathfrak{c})(\mathbf{N}b[\beta]^* - 1)\Theta^{dR}(n)|U = (\mathbf{N}\mathfrak{c}[\epsilon]^* - \mathbf{N}\mathfrak{c})_{\mathfrak{b}}\Theta^{dR}(\alpha) = (\mathbf{N}\mathfrak{b}[\beta]^* - 1)^{\mathfrak{c}}\Theta^{dR}(n) = {}^{\mathfrak{c}}_{\mathfrak{b}}\Theta^{dR}(n) \tag{4.15} $$

if $\mathfrak{c} = (\epsilon)$, $\mathfrak{b} = (\beta)$ are generated by positive-norm elements.

We are now ready to define the stabilized conical duality map and prove its properties. For convenience, let us denote the inverse transpose involution on $\mathrm{GL}_n(\mathbb{Q})$ by $\star$. Then $(\star^*)^b_c\mathrm{St}(n)$ carries the action

$$([\ell_1] \wedge \ldots \wedge [\ell_n])_{\eta,\lambda} \mapsto ([(\gamma^T)^{-1}\ell_1] \wedge \ldots \wedge [(\gamma^T)^{-1}\ell_n])_{(\gamma^T)^{-1}\eta,(\gamma^T)^{-1}\lambda}.$$

Specializing this, we also write $(\star^*)^{\mathfrak{b}}_{\mathfrak{c}}\mathrm{St}(\alpha)$ for the $\star U$-module with the same underlying space as ${}^{\mathfrak{b}}_{\mathfrak{c}}\mathrm{St}(\alpha)$, on which $\gamma \in \star U$ acts by the action of $\star\gamma \in U$ on ${}^{\mathfrak{b}}_{\mathfrak{c}}\mathrm{St}(n)$.

Proposition 4.18. *There is a $\mathrm{GL}_n(\mathbb{Z})$-equivariant map*

$${}^c_b\delta : {}^c_b\mathrm{St}(n) \to (\star^*)^b_c\mathrm{St}(n)$$

given by sending

$$(([\ell_1] \wedge \ldots \wedge [\ell_n]))_{\eta,\lambda} \mapsto ([\ell_1^\vee] \wedge \ldots \wedge [\ell_n^\vee])_{\lambda^\perp,\eta^\perp}.$$

The same formula yields a U-equivariant map

$${}^{\mathfrak{c}}_{\mathfrak{b}}\delta : {}^{\mathfrak{c}}_{\mathfrak{b}}\mathrm{St}(\alpha) \to {}^{\mathfrak{b}}_{\mathfrak{c}}\mathrm{St}(\alpha^\star)$$

when $\mathfrak{b}, \mathfrak{c}$ are ideals in F, where

$$\alpha^\star : \mathfrak{d}^{-1}I^{-1} \xrightarrow{\sim} (\mathbb{Z}^n)^\vee$$

is the $(\star\iota_\alpha)(U)$-equivariant map

$$\mathfrak{d}^{-1}I^{-1} \xrightarrow{\sim} I^\vee \xrightarrow{(\alpha^{-1})^\vee} \mathbb{Z}^n \xrightarrow{v\mapsto\langle v,-\rangle} (\mathbb{Z}^n)^\vee$$

where the first arrow is from the trace pairing on F. Note that here, the U-action on the lines in the source is defined via ι_α as before, whereas the action on the lines in the target is via $\iota_{\alpha^\star} = \star\iota_\alpha$.[13]

Proof. The proof in the U-equivariant case is formally identical to the general linear case, so we treat just the latter.

If the formulas are well-defined, then $\mathrm{GL}_n(\mathbb{Z})$- (respectively U-)equivariance are clear. We have already seen previously that the Steinberg module is the quotient of $\widetilde{\mathrm{Chains}}(n)_n$ by wedges, and the latter quotient

$$\mathcal{K}_{\mathbb{Q}}/\mathcal{L}_{\mathbb{Q}} \to \mathrm{St}(n)$$

is given on top-dimensional generators as

$$1_{\mathbb{R}^+\Delta^\circ(r_1,\ldots,r_n)} \mapsto [\ell_1] \wedge \ldots \wedge [\ell_n].$$

This is well-defined by the earlier discussion of "M-additivity" (along with the evident fact that there can be no relations between a top-dimensional cone indicator function and any combination of lower-dimensional ones). One then needs only to note that the "local" conditions with respect to a c-torsion line λ and a d-hyperplane η are dual to each other under the standard inner product, and hence under δ: indeed, if $[\ell_1] \wedge \ldots \wedge [\ell_n]$ avoids λ, then the line of c-torsion characters given by taking the standard inner product with λ precisely have kernels $\lambda^\perp$, and are hence nontrivial on each of $\ell_1^\vee, \ldots, \ell_n^\vee$: the former condition says none of the $(n-1)$-hyperplanes spanned by these pass through λ, which is the same as saying that the line λ is not in the kernel of the any of the dual linear forms $\langle \ell_\bullet^\vee, -\rangle$, which is to say that none of these lines were contained in $\lambda^\perp$ to begin with. The η conditions are symmetric. ∎

[13]In particular, the definitions of the modules in question only depend on the algebra embedding $\iota_\alpha : F \to M_n(\mathbb{Q})$ and not the actual coordinatization α; the map we denote $\alpha^\star$ is just a particular canonical-looking such coordinatization whose corresponding algebra embedding is $\star\iota_\alpha(F)$-equivariant. For the purposes of this proposition, however, it is not any better or worse than any other coordinatization in its F-isogeny class.

We consider now the classes

$$ {}^{c}_{b}\delta_{*}{}^{c}_{b}\Theta^{St}(n) \in H^{n-1}(\mathrm{GL}_n(\mathbb{Z}), (\star^*){}^{b}_{c}\mathrm{St}(n)), \quad {}^{\mathfrak{c}}_{\mathfrak{b}}\delta_{*}{}^{\mathfrak{c}}_{\mathfrak{b}}\Theta^{St}(\alpha) \in H^{n-1}(U, {}^{\mathfrak{b}}_{\mathfrak{c}}\mathrm{St}(\alpha^\star)) $$

whose realizations under ${}^{b}_{c}r_*$ and ${}^{\mathfrak{b}}_{\mathfrak{c}}r_*$ we notate respectively as

$$ {}^{b}_{c}\Theta^{Shin}(n) \in H^{n-1}(\mathrm{GL}_n(\mathbb{Z}), (\star^*)A_p(\mathrm{sgn})^{(n)}), \quad {}^{\mathfrak{b}}_{\mathfrak{c}}\Theta^{Shin}(\alpha) \in H^{n-1}(U, A_p(\mathrm{sgn})^{(n)}_{\star\iota_\alpha}), \tag{4.16} $$

where we have put a subscript to record the action of U on A_p.

To prove Theorem 4.16, we will first need a statement to the effect that ${}^{c}_{b}\delta$ "acts trivially" on cohomology: we note that $\star$ induces a kind of tautological action on $\mathrm{GL}_n(\mathbb{Z})$-cohomology

$$ \star^* : H^i(\mathrm{GL}_n(\mathbb{Z}), M) \to H^i(\mathrm{GL}_n(\mathbb{Z}), (\star^*)M) $$

by acting on cochains as $c(\gamma_1, \ldots, \gamma_i) \mapsto (c \circ \star^i)(\gamma_1, \ldots, \gamma_i)$.

Then the main part of the result we need is a considerable technical digression which we defer to the appendix:

Proposition 4.19. *The class ${}^{b}_{c}\Theta^{Shin}(n)$ is equal to $(\star^*){}^{b}_{c}\Theta^{dR}(n)$, and correspondingly, ${}^{\mathfrak{b}}_{\mathfrak{c}}\Theta^{Shin}(\alpha)$ is equal to ${}^{\mathfrak{b}}_{\mathfrak{c}}\Theta^{dR}(\alpha^\star)$.*

Proof. Using a duality of (restricted) Tits buildings giving rise to (restricted) Steinberg representations in their top reduced homology, we prove in Appendix B that ${}^{c}_{b}\delta_{*}{}^{c}_{b}\Theta^{St}(n)$ is equal to ${}^{b}_{c}\Theta^{St}(n)$, which implies the proposition by taking realizations. The statement for $\mathfrak{b}$ and $\mathfrak{c}$ follows similarly from the equality of ${}^{\mathfrak{c}}_{\mathfrak{b}}\delta_{*}{}^{\mathfrak{c}}_{\mathfrak{b}}\Theta^{St}(\alpha)$ and ${}^{\mathfrak{b}}_{\mathfrak{c}}\Theta^{St}(\alpha^\star)$, proved in the same way. ☐

4.4. **Results on L-values.**

4.4.1. *Interpolation property for smoothed/avoiding cocycles.* We now compute the values of ${}^{\mathfrak{b}}_{\mathfrak{c}}\Theta^{Shin}(\alpha^\star)$ using the Shintani method. The notation in the following proposition is as in Section 4.3.4.

Proposition 4.20. *Let $\mathfrak{b}$ and $\mathfrak{c}$ be ideals as previously. We identify $\kappa_*{}^{\mathfrak{b}}_{\mathfrak{c}}\Theta^{Shin}(\alpha^\star)$ with a class in $H^{n-1}(U^+, \mathbf{D}(F_p, \mathbb{Z}_p)^{(0)})$ via the map (4.9). (Note that the natural action of U on the space F_p, identified via α with $(\mathbb{Z}^n)^\vee$, in the distributions is via ι_α, by preceding discussion.)*

Then the image of this class under the chain of arrows (4.11) (which we will also denote ${}^{\mathfrak{b}}_{\mathfrak{c}}\zeta_p^{I,\alpha}$, in analogy with ${}_{\mathfrak{c}}\zeta_p^{I,\alpha}$), when evaluated against the function

$$ F_p/U^+ \to \overline{\mathbb{Q}_p}^\times, t \mapsto \psi(t)\mathbf{N}(t)^k $$

for any Schwartz function $\psi : I_p/U^+ \to \overline{\mathbb{Q}_p}^\times$, is

$$ (-1)^{n-1}\mathbf{N}b(\zeta_I(\psi_{\mathfrak{b}} - \psi, -k) - \mathbf{N}\mathfrak{c}\zeta_{\mathfrak{c}I}(\psi_{\mathfrak{b}} - \psi, -k)) $$

where $\psi_{\mathfrak{b}} : I/U^+ \to \overline{\mathbb{Q}_p}^\times$ is the function $t \mapsto 1_{\mathfrak{b}I}(t)\psi(t)$ on I. (Note that this cannot be defined on I_p/U^+.)

Proof. We recall the definition of ${}^{\mathfrak{b}}_{\mathfrak{c}}\Theta^{Shin}(\alpha^\star)$ is

$$ ({}^{\mathfrak{b}}_{\mathfrak{c}}r)_*({}^{\mathfrak{c}}_{\mathfrak{b}}\delta)_*{}^{\mathfrak{c}}_{\mathfrak{b}}\Theta^{St}(\alpha^\star). \tag{4.17} $$

Let $U(\mathfrak{b}, \mathfrak{c})^+ \subset U^+$ be the congruence subgroup reducing to the identity modulo $\mathfrak{b}$ and $\mathfrak{c}$. This is a finite-index subgroup of the rank-$(n-1)$ free abelian group U^+; let $c^\alpha_{\mathfrak{b},\mathfrak{c}}$ be the positively-oriented fundamental class of $U(\mathfrak{b}, \mathfrak{c})^+$, so that $c^\alpha_{\mathfrak{b},\mathfrak{c}}$ pushes forward to $[U^+ : U(\mathfrak{b},\mathfrak{c})^+]c^{\alpha}_{U^+}$ under the subgroup inclusion. Then we compute (4.17) as

$$\frac{1}{[U^+ : U(\mathfrak{b},\mathfrak{c})^+]}\mathrm{cores}^{U^+}_{U(\mathfrak{b},\mathfrak{c})^+}\,\mathrm{res}^{U^+}_{U(\mathfrak{b},\mathfrak{c})^+}({}^{\mathfrak{b}}_{\mathfrak{c}}r)_*({}^{\mathfrak{c}}_{\mathfrak{b}}\delta)_*{}^{\mathfrak{c}}_{\mathfrak{b}}\Theta^{St}(\alpha^\star). \tag{4.18}$$

We spell out the inside symbol $\Xi := \mathrm{res}^{U^+}_{U(\mathfrak{b},\mathfrak{c})^+}({}^{\mathfrak{b}}_{\mathfrak{c}}r)_*({}^{\mathfrak{c}}_{\mathfrak{b}}\delta)_*{}^{\mathfrak{c}}_{\mathfrak{b}}\Theta^{St}(\star\alpha^\star)$. Let $u_1,\dots,u_{n-1}$ be a basis of $U(\mathfrak{b},\mathfrak{c})^+$, which we will implicitly view via $\star\iota_\alpha$ inside $\mathrm{GL}_n(\mathbb{Z})$. We write T_1 for $\mathbb{G}_m^n$ considered with the ι_α-action, and T_2 for $\mathbb{G}_m^n$ considered with the $\star\iota_\alpha$-action; the map of Proposition 4.18 then U-equivariantly identifies the torsion of T_1 with the Pontryagin dual of torsion of T_2 of the same order, and vice versa.

We compute

$$\begin{aligned}\Xi(u_1,\dots,u_{n-1}) &= {}^{\mathfrak{b}}_{\mathfrak{c}}r_*\left(-\sum_{\lambda_1\in\mathbb{P}(T_2[\mathfrak{c}])}\ \sum_{\lambda_2\in\mathbb{P}(T_2[\mathfrak{b}])}[u_1\dots u_{n-1}\ell_{\lambda_1,\lambda_2^\perp},\dots,\ell_{\lambda_1,\lambda_2^\perp}]\right)\\ &= -\sum_{\xi\in\widetilde{T_2[\mathfrak{c}]}-\{1\}}\ \sum_{y\in T_2[\mathfrak{b}]-\{1\}}\ \sum_{\chi\in\widetilde{T_2[b]}}\chi(y)f_{C(\mathbb{Z}/c\cdot\xi^\perp,(\mathbb{Z}/b\cdot y)^\perp)}(\xi^{-1}\cdot\chi)\in A_p\end{aligned}$$

where $\ell_{\lambda,\eta}$ is any choice of ray satisfying suitable avoidance conditions for λ and η, and $C(\lambda,\eta)$ the associated cone bounded by the U-translates indicated.[14] From (4.5), Theorem 4.4, and Proposition 4.5, we then find that the realization under $\widetilde{\kappa}$ of Ξ satisfies

$$\int_{(\mathbb{Z}_p)^\vee}\psi(\alpha^{-1}(t))\mathbf{N}(\alpha^{-1}(t))^k\,d\kappa(\Xi\frown c^{\alpha}_{\mathfrak{b},\mathfrak{c}})(t) = (-1)^n\sum_{\lambda\in I_p^+/U(\mathfrak{b},\mathfrak{c})^+}\ \sum_{\xi\in\widetilde{T_2[\mathfrak{c}]}-\{1\}}\ \sum_{y\in T_2[\mathfrak{b}]-\{1\}}\ \sum_{\chi\in\widetilde{T_2[\mathfrak{b}]}}\chi(y)\chi(\lambda)\xi^{-1}(\lambda)\mathbf{N}\lambda^k \tag{4.19}$$

$$= (-1)^{n-1}[U^+ : U(\mathfrak{b},\mathfrak{c})^+]\mathbf{N}\mathfrak{b}(\zeta_I(\psi_{\mathfrak{b}}-\psi,-k)-\mathbf{N}\mathfrak{c}\zeta_{\mathfrak{c}I}(\psi_{\mathfrak{b}}-\psi,-k)) \tag{4.20}$$

The last equality follows from Shintani's Theorem 4.4 because all the terms corresponding to any lower-dimensional conical face C' vanish from the $\mathfrak{b}$-stabilization in the sum: there exists a line in the dual space (uniformizing T_1) annihilating the subspace $\mathbb{Q}\cdot C'\subset\mathbb{R}^n$ (uniformizing T_2); taken modulo b, this means there is a line of $\chi\in T_1[b]$ for which $f_{C'} = f_{C'}(\chi)$. The terms $f_{C'}(\xi^{-1}\cdot\chi)$ thus are all equal on cosets for this line in the sum over χ. Then y, considered as a character on these χ, is nontrivial on this line, due to our initial assumption that (in the dual formulation) y is not contained in any of the implicated lines modulo b (cf. the discussion before Example 4.1). Thus, the coefficients in the total sum on each coset of these lines sums to zero, whence all the terms corresponding to generating functions of C' vanish.

The result then follows from the commutative diagram in group (co)homology

$$\begin{tikzcd} H^{n-1}(U(\mathfrak{b},\mathfrak{c})^+,\mathbf{D}(F_p,\mathbb{Z}_p)^{(0)}) \arrow[r, "\frown c^{\alpha}_{\mathfrak{b},\mathfrak{c}}"] \arrow[d, "\mathrm{cores}"] & H_0(U(\mathfrak{b},\mathfrak{c})^+,\mathbf{D}(F_p,\mathbb{Z}_p)^{(0)}) \arrow[r] \arrow[d, "\mathrm{cores}"] & \mathbf{D}(F_p/U(\mathfrak{b},\mathfrak{c})^+,\mathbb{Z}_p)^{(0)} \arrow[d, "\text{projection}"] \\ H^{n-1}(U^+,\mathbf{D}(F_p,\mathbb{Z}_p)^{(0)}) \arrow[r, "\frown c^{\alpha}_{U^+}"] & H_0(U^+,\mathbf{D}(F_p,\mathbb{Z}_p)^{(0)}) \arrow[r] & \mathbf{D}(F_p/U^+,\mathbb{Z}_p)^{(0)} \end{tikzcd} \tag{4.21}$$

□

Remark 4.21. We note that working with the $\mathrm{GL}_n(\mathbb{Z})$-version of the duality statement was not logically necessary for the proof of this theorem: only the U-restricted duality, with the more general stabilizations at ideals $\mathfrak{b},\mathfrak{c}\subset F$, is strictly required. However,

[14] In the simpler case when $\mathfrak{b}$ and $\mathfrak{c}$ have residual degree 1, so that the corresponding torsion subgroups are actually lines, one can actually find one line ℓ which works across the board; i.e. we can compute Ξ simply as ${}^{\mathfrak{b}}_{\mathfrak{c}}r_*\left(-[(u_1\dots u_{n-1}\ell)^\vee,\dots,(u_1\ell)^\vee,\ell^\vee]_{\lambda_{\mathfrak{c}}^\perp,\lambda_{\mathfrak{b}}}\right)$ for some appropriate line ℓ. This is the kind of stabilization usually considered in the literature (e.g. in [CDG]), since it results in simpler-to-compute formulas whose existence can be deduced without the symbolic cohomological considerations of the present article.

we find the duality statement (and the geometry behind its proof in Appendix B) much clearer to understand from the $\mathrm{GL}_n(\mathbb{Z})$-equivariant perspective, because of the essential intervention of its involution $\star$ (which simply "switches the embedding" when acting on U, in a potentially confusing way if one is not keeping track of the ambient group). Thus, we emphasized its role in the exposition.

4.4.2. *Assembling the partial zeta functions.* To deduce Theorem 4.16 from our above calculations, and in particular, to remove all $\mathfrak{b}$-smoothing, we need to assemble the various different partial zeta functions into one framework. Recall the formalism of L-functions associated to the narrow Hilbert class group of F: following [Katz], given an integral ideal $\mathfrak{N} \subset F$, we write $M_{\mathfrak{N}}$ for the monoid (under multiplication) of integral ideals of $\mathcal{F}$, modulo the equivalence relation $I \sim J$ whenever IJ^{-1} is generated by a totally positive element in $1 + \mathfrak{N}J^{-1}$. We consider also the inverse limit

$$M_{p^\infty} := \varprojlim_k M_{p^k}.$$

As noted previously, we will omit consideration of the more general construction with tame level $M_{\mathfrak{N}(p^\infty)}$ considered in [Katz]; however, the same results should follow from identical arguments.

Let G_{p^k} be the p^k-ray class group of F, and $G_{p^\infty} := \varprojlim G_{p^k}$; it acts on M_{p^∞} by level-wise multiplication. Write also Σ for the subgroup of G_{p^∞} generated by towers of principal ideals generated by elements $1 \pmod{p^k}$; writing S_∞ for the set of infinite places of F, we have a canonical isomorphism $\Sigma \cong \{\pm 1\}^{S_\infty}$, the generator at the place ν given by a tower of principal ideals which are negative at precisely ν and positive at all the other places. We also have a norm map $\mathbf{N} : \Sigma \to \{\pm 1\}$ by taking the product over all places. We say the a function $\varphi : M_{p^\infty} \to \overline{\mathbb{Q}_p}$ has *sign* $\operatorname{sgn}(\varphi) \in \{\pm 1\}^{S_\infty}$ if $\varphi \circ \sigma = \operatorname{sgn}(\varphi) \cdot \varphi$ for all $\sigma \in \Sigma$; if $\operatorname{sgn}(\varphi)$ consists of all -1s, we call φ totally odd, and if $\operatorname{sgn}(\varphi)$ is all 1s, we call it totally even.

From [DR, §2], for any $k \geqslant 0$, the finite-level ideal class monoid M_{p^k} can canonically be decomposed as a disjoint union of "cells" which are ray class groups of conductors dividing p^k, via the Σ-equivariant isomorphism

$$\mathcal{D}_k : \bigsqcup_{\mathfrak{d}|(p^k)} G_{(p^k)\mathfrak{d}^{-1}} \xrightarrow{\sim} M_{p^k}, [J]_{(p^k)\mathfrak{d}^{-1}} \mapsto [\mathfrak{d} \cdot J]_{(p^k)}$$

One can compute that the monoid multiplication between two "cells" corresponding to $(p^k)\mathfrak{d}_1^{-1}$ and $(p^k)\mathfrak{d}_2^{-1}$ lands in the "cell" corresponding to their GCD, and is given by usual ideal multiplication on representatives. All of this can be extended to infinite level M_{p^∞} in the obvious way by taking inverse limits everywhere, yielding

$$\mathcal{D}_\infty : G_1 \sqcup \bigsqcup_{\mathfrak{d}|(p^\infty)} G_{p^\infty/\mathfrak{d}} \xrightarrow{\sim} M_{p^\infty}$$

where each $G_{p^\infty/\mathfrak{d}}$ is abstractly just a copy of the p^∞-ray class group, with the different labels just serving to distinguish them. Note that the copy of G_1 is, however, not open: in the case $F = \mathbb{Q}$, this is the decomposition (up to ± 1)

$$\mathbb{Z}_p = \{0\} \sqcup \bigsqcup_{j=0}^{\infty} p^j \mathbb{Z}_p^\times.$$

Choose now integral representatives $I_1, \dots, I_h$ for the narrow Hilbert ideal classes of F. For each $1 \leqslant i \leqslant h$, fix an isomorphism

$$\alpha_i : I_i \xrightarrow{\sim} (\mathbb{Z}^n)^\vee$$

and a corresponding embedding ι_{α_i}, along with the fundamental class $c_{U^+}^{\alpha_i}$ satisfying the condition of Definition 4.10.

We have the identification

$$f : \bigsqcup_{i=1}^{h} \left(\varprojlim I_i/p^k I_i\right)/U^+ \xrightarrow{\sim} M_{p^\infty}, (x_k)_k \mapsto ((x_k)\cdot I_i^{-1})_k, \text{ for } (x_k)_k \in \varprojlim I_i/p^k I_i$$

from loc. cit. The norm function on ideals extends to a continuous map $\mathbf{N} : M_{p^\infty} \to \mathbb{Z}_p$, and we have the relation

$$(\mathbf{N}\circ f)(x) = \mathbf{N}I_i^{-1}\cdot \mathbf{N}x$$

for $x \in I_i \otimes \mathbb{Z}_p$. Given a locally constant function $\psi : M_{p^\infty} \to \mathbb{C}$, we have the associated complex L-function analytically continuing the Dirichlet series

$$\zeta_F(\psi, s) := \sum_J \frac{\psi([J])}{(\mathbf{N}J)^s}$$

where the sum ranges over all integral ideals of F. From ψ, for each class $[I_i]$, we also get an associated locally constant function

$$\psi_i : (I_i)_p \to \overline{\mathbb{Q}_p}, t \mapsto \psi(f(t)).$$

for which we can form $\zeta_{I_i}(\psi_i, s)$ as previously; then we have, from the identification coefficient-wise of the associated Dirichlet series, that

$$\zeta_F(\psi, s) := \sum_{i=1}^{h} (\mathbf{N}I_i)^s \zeta_{I_i}(\psi_i, s).$$

For any $1 \leqslant i \leqslant h$, write also T_i for $\mathbb{G}_m^n$ with the action corresponding to α_i under the embedding ι_i, i.e. so that the character lattice of T_i is identified with I_i under α_i, as U-representations where U is viewed as a subgroup of $\mathrm{GL}_n(\mathbb{Z})$ via ι_i. As U-spaces, we have $T_i = \hom(I_i, \mathbb{G}_m)$.

We note also that if $\psi : G_{p^\infty} \to \mathbb{C}$ is locally constant, and we consider it as a function on $G_{p^\infty/\mathfrak{d}}$, then $(\mathcal{D}_\infty^{-1})^*\psi$ is a function on M_{p^∞}, and one can check that

$$\zeta_F(\psi, s) = \mathbf{N}\mathfrak{d}^s \zeta_F((\mathcal{D}_\infty^{-1})^*\psi, s).$$

One can also express this relation at each finite level p^k.

Suppose we have any relatively prime ideals $\mathfrak{b}, \mathfrak{c}$ both prime to p, and that $[I_i\mathfrak{b}] = [I_j]$ for some $1 \leqslant i, j \leqslant h$; thus, we can write $\varpi\mathcal{O}_F = I_i\mathfrak{b}I_j^{-1}$ for some totally positive $\varpi \in F_+^\times$. Fix also a positive integer $t \in \mathbb{N}$ for which $\varepsilon := t\varpi \in \mathcal{O}_F^+$. For a judicious choice of representatives $I_\bullet$, we can always ensure that the resulting t can be chosen prime to p and $\mathfrak{c}$, for any $\mathfrak{b}$ prime to p and $\mathfrak{c}$.

Then there is a canonical U-equivariant identification

$$T_j/T_j[\varepsilon] \cong T_i/T_i[t\mathfrak{b}]$$

dual to the identification $(t)I_i\mathfrak{b} = \varepsilon I_j$ of character lattices. We notate below several related U-equivariant maps:

$$T_j \xleftarrow{[\varepsilon]} T_j/T_j[\varepsilon] = T_i/T_i[t\mathfrak{b}] \xrightarrow{[t]} T_i/T_i[\mathfrak{b}] \xleftarrow{q_\mathfrak{b}} T_i, \tag{4.22}$$

these maps all being isomorphisms except for $q_\mathfrak{b}$, of degree $\mathbf{N}\mathfrak{b}$. The dual maps of character lattices are

$$I_j \xrightarrow{\times\varepsilon} I_j(\varepsilon) = I_i(t)\mathfrak{b} \xleftarrow{\times t} I_i\mathfrak{b} \hookrightarrow I_i.$$

Now write ${}_\mathfrak{c}\zeta_p^i := {}_\mathfrak{c}\zeta_p^{I_i,\alpha_i}$ for $1 \leqslant i \leqslant h$, and analogously for the $\mathfrak{b}$-smoothed versions. Similarly, write ${}_\mathfrak{c}\Theta_i^{dR} := {}_\mathfrak{c}\Theta^{dR}(\alpha_i)$ and analogously for the $\mathfrak{b}$-smoothed versions. Then we have:

Proposition 4.22. *We have the relation*

$${}^{\mathfrak{b}}_{\mathfrak{c}}\zeta_p^i = \mathbf{N}\mathfrak{b}([\varpi]_* {}_\mathfrak{c}\zeta_p^j - {}_\mathfrak{c}\zeta_p^i)$$

where we write $[\varpi]_*$ for the map

$$\mathbf{D}((I_j)_p, \mathbb{Z}_p)^{(0)} \to \mathbf{D}((I_i)_p, \mathbb{Z}_p)^{(0)}, \mu \mapsto (\varphi \mapsto \mu(\varphi \circ [\varpi])).$$

Proof. It suffices to prove that

$$ {}^{\mathfrak{b}}_{\mathfrak{c}}\Theta^i_p = \mathbf{N}\mathfrak{b}(q^*_{\mathfrak{b}}([t]^*)^{-1}[\varepsilon]^* {}_{\mathfrak{c}}\Theta^j_p - {}_{\mathfrak{c}}\Theta^i_p) \in H^{n-1}(U, A^{(n)}_p(T_i)) \tag{4.23}$$

since by the pullback functoriality (4.2), the action of (e.g.) $[\varepsilon]_*$ on meromorphic functions is $[\varepsilon]^*$ on distributions, recalling that the pullback action on $(\mathbb{Z}^n)^\vee$ becomes the natural action of F on F_p. Note that here, we decorate A_p with the space on which it is considered, since this is no longer simply implicit. Observe also that when $\mathfrak{b}$ is principal and generated by an element of positive norm, this is consistent with our earlier stabilization statements.

We have a U-equivariant map of complexes which we denote with the notation

$$\{\varpi\} : {}_{\mathfrak{c}}\mathrm{Chains}(n)^{\alpha_j,+}_\bullet \to \bigoplus_{\rho \in \mathbb{P}(T_i[\mathfrak{b}])} {}_{\mathfrak{c}}\mathrm{Chains}(n)^{\alpha_i,+}_\bullet$$

defined by sending $\Delta(r_1, \ldots, r_k)_\lambda$ (for $\lambda \in \mathbb{P}(T_j[\mathfrak{c}])$) to

$$\sum_{\rho \in \mathbb{P}(T_i[\mathfrak{b}])} [(\alpha_i\alpha_j^{-1})_*\Delta(r_1, \ldots, r_k)]_{\lambda',\rho}$$

where $\lambda' \in \mathbb{P}(T_i[\mathfrak{c}])$ is the image of λ coming from the series of maps

$$T_j[\mathfrak{c}] \cong \hom_{\mathcal{O}_F}(I_j, \mathcal{O}_F/\mathfrak{c}) \xleftarrow{[\varepsilon]^{-1}} \hom_{\mathcal{O}_F}(\varepsilon I_j, \mathcal{O}_F/\mathfrak{c}) = \hom_{\mathcal{O}_F}(\mathfrak{b}(t)I_i, \mathcal{O}_F/\mathfrak{c}) \xrightarrow{[t]} \hom_{\mathcal{O}_F}(I_i, \mathcal{O}_F/\mathfrak{c}) = T_i[\mathfrak{c}] \tag{4.24}$$

attached to (4.22).

In top degree, we claim this leads to a commutative diagram

$$\begin{tikzcd} {}_{\mathfrak{c}}\mathrm{Chains}(n)^{\alpha_j,+}_n \arrow[r, "\{\varpi\}"] \arrow[d, "{}_{\mathfrak{c}}r"] & \bigoplus_{y \in T_i[\mathfrak{b}]} {}_{\mathfrak{c}}\mathrm{Chains}(n)^{\alpha_i,+}_n \arrow[d, "{}^{\mathfrak{b}}_{\mathfrak{c}}r"] \\ A_p(T_j)^{(n)} \arrow[r, "\mathbf{N}\mathfrak{b}\cdot q^*_{\mathfrak{b}}\circ([t]^{-1})^*\circ[\varepsilon]^*"] & A_p(T_i)^{(n)} \end{tikzcd} \tag{4.25}$$

where here we reuse the notation ${}^{\mathfrak{b}}_{\mathfrak{c}}r$ to denote sending $\Delta(r_1, \ldots, r_n)_{\lambda,\rho}$ to

$$\sum_{x \in \lambda - \{1\}} \sum_{y \in \rho - \{1\}} t^*_{y+x} \begin{pmatrix} r_1 & \ldots r_n \end{pmatrix}_* \frac{z_1 \ldots z_n}{(z_1 - 1) \ldots (z_n - 1)}.$$

(One checks this compatibly extends the earlier definition of ${}^{\mathfrak{b}}_{\mathfrak{c}}r$ on the submodule ${}^{\mathfrak{b}}_{\mathfrak{c}}\mathrm{Chains}(n)^\bullet_n$, considering characters on $T_i[\mathfrak{b}]$ as formal sums over these torsion points.)

To verify the commutativity, by continuity it suffices to verify it after specialization (of $A_p(T_i)^{(n)}$) at any torsion point

$$z \in T_i[p^\infty] \cong \varinjlim \hom(I_i, \mu_{p^k})$$

(In other words, the bottom vertical arrows are injective, and one can identify distributions by evaluating against finite-order characters of lattices.) Indeed, via the top right chain of arrows composed with the specialization, the image of $\Delta(r_1, \ldots, r_n)_x$ is

$$z^* {}^{\mathfrak{b}}_{\mathfrak{c}}r[(\alpha_i\alpha_j^{-1})_*\Delta(r_1, \ldots, r_n)_x] = \mathbf{N}\mathfrak{b} \sum_{y \in T_i[\mathfrak{b}]} (z' - x' - y)^* \frac{z_1 \ldots z_n}{(z_1 - 1) \ldots (z_n - 1)}$$

(where we are writing additively the group law on T_i), where z' is the image of z along the map on p^∞-torsion analogous to (4.24); this agrees with the image along the bottom chain. Note the factor $\mathbf{N}\mathfrak{b} = \deg(q_\mathfrak{b} \circ [t]^{-1} \circ [\varepsilon])$ comes from the determinant twist resulting from contracting away the form $dt_1 \wedge \ldots \wedge dt_n$, as previously.

Let $\{1\}_{\mathfrak{c},\bullet} := T_\bullet[\mathfrak{c}] - \{1\} \in \mathbb{Z}\{T_\bullet[\mathfrak{c}]\}$, for $\bullet \in \{1, 2, \ldots, h\}$. Since ${}^\mathfrak{b}_\mathfrak{c}\Theta^{dR}_i(\alpha)$ can be obtained via lifting the class

$$\{\varpi\}\{1\}_{\mathfrak{c},i} - \mathbf{N}\mathfrak{b} \cdot \{1\}_{\mathfrak{c},j} \in \mathbb{Z}\{T_i[\mathfrak{b}\mathfrak{c}]\}$$

in degree zero of the complex $\bigoplus_{y \in T_i[\mathfrak{b}]} {}_\mathfrak{c}\mathrm{Chains}(n)^{\alpha_i,+}_\bullet$, (4.23) follows. □

Now, the map f induces a pushforward on distributions, which preserves the $[p]$-invariance property; i.e. yields

$$f_* : \bigoplus_{i=1}^{h} \mathbf{D}(I_p/U^+, \mathbb{Z}_p)^{(0)} \to \mathbf{D}(M_{p^\infty}, \mathbb{Z}_p)^{(0)}$$

where, analogously to before, the superscript (0) on distributions on the monoid means that $\mu(U) = \mu((p) \cdot U)$ for any open compact U. Note that under the cell decomposition, multiplication by (p) maps $G_{p^\infty/\mathfrak{d}} \to G_{p^\infty/(p)\mathfrak{d}}$.

Accordingly, we now define p-adic L-elements

$$\mathbf{D}(M_{p^\infty}, \mathbb{Z}_p)^{(0)} \ni {}_\mathfrak{c}\zeta_p^F := \sum_{i=1}^{h} f_* {}_\mathfrak{c}\zeta_p^{I_i,\alpha_i}, \quad \mathbf{D}(M_{p^\infty}, \mathbb{Z}_p)^{(0)} \ni {}^\mathfrak{b}_\mathfrak{c}\zeta_p^F := \sum_{i=1}^{h} f_* {}^\mathfrak{b}_\mathfrak{c}\zeta_p^{I_i,\alpha_i} \tag{4.26}$$

Now, applying Pontryagin duality to each cell of the cell decomposition of M_{p^∞} given by $\mathcal{D}_\infty$ implies that Schwartz functions on M_{p^∞} are spanned by orbits under multiplication-by-(p) of the following two forms of functions:

(1) *ramified* finite-order characters of G_{p^∞} (included as any cell), meaning which do not factor through $G_{p^\infty} \to G_1$, and
(2) unramified characters, i.e. those factoring through the projection $M_{p^\infty} \to M_1 = G_1$.

For any character of either form above $\psi : M_{p^\infty} \to \overline{\mathbb{Q}}$, Proposition 4.20 then implies the interpolation property

$$\int_{M_{p^\infty}} \psi(J)\mathbf{N}J^k \, d{}^\mathfrak{b}_\mathfrak{c}\zeta_p^F(J) = \sum_{i=1}^{h} \mathbf{N}I_i^{-k} \int_{(I_i)_p/U^+} \psi_i(t)\mathbf{N}t^k \, d{}^\mathfrak{b}_\mathfrak{c}\zeta_p^F(t) \tag{4.27}$$

$$= (-1)^{n-1}\mathbf{N}\mathfrak{b} \sum_{i=1}^{h} \mathbf{N}I_i^{-k}(\zeta_{I_i}(\psi_i - (\psi_i)_\mathfrak{b}, -k) - \mathbf{N}\mathfrak{c}\zeta_{\mathfrak{c}I_i}((\psi_i)_\mathfrak{b} - \psi_i, -k)) \tag{4.28}$$

$$= (-1)^{n-1}\mathbf{N}\mathfrak{b}(1 - \psi([\mathfrak{c}])\mathbf{N}\mathfrak{c}^{1+k})\zeta_F(\psi_\mathfrak{b} - \psi, -k) \tag{4.29}$$

This suffices to determine the interpolation properties for all Schwartz functions on M_{p^∞}, since our zeta distributions are invariant under multiplication by (p). Note that the multiplicativity of ψ is only needed at the last line; we could have also have written the interpolation property for a general Schwartz function, but for these spanning characters, it has a more compact form thanks to this factorization.

Proof of Theorem 4.16. We have, for any ideal $\mathfrak{b}$ in F prime to p and $\mathfrak{c}$, that ${}^\mathfrak{b}_\mathfrak{c}\Theta^{dR}(\alpha) = {}^\mathfrak{b}_\mathfrak{c}\Theta^{Shin}(\alpha^\star)$. From Proposition 4.20, recall that the value of this cocycle at $u_1, \ldots, u_{n-1}$, after applying (4.11), satisfies

$$\int_{(\mathbb{Z}_p^n)^\vee} \psi(\alpha^{-1}(t))\mathbf{N}(\alpha^{-1}(t))^k \, d\kappa({}^\mathfrak{b}_\mathfrak{c}\Theta^{dR}(\alpha) \frown c^\alpha_{U^+})(t) = (-1)^{n-1}\mathbf{N}\mathfrak{b}(\zeta_I(\psi_\mathfrak{b} - \psi, -k) - \mathbf{N}c \cdot \zeta_{\mathfrak{c}I}(\psi_\mathfrak{b} - \psi, -k))$$

for any locally constant ψ as previously. Then it results from Proposition 4.22 that for any $\mathfrak{b}, i, j, \varpi$ as in the proposition statement, we have

$$ {}_{\mathfrak{c}}\zeta_p^{I_i,\alpha_i}(\psi \cdot \mathbf{N}^k) - [\varpi]^* {}_{\mathfrak{c}}\zeta_p^{I_j,\alpha_j}(\psi \cdot \mathbf{N}^k) = (-1)^{n-1}\mathbf{N}\mathfrak{b}(\zeta_I(\psi_{\mathfrak{b}} - \psi, -k) - \mathbf{N}\mathfrak{c}\zeta_{\mathfrak{c}I}(\psi_{\mathfrak{b}} - \psi, -k)) \tag{4.30} $$

for any locally constant ψ, and any integer $k \geqslant 0$.

From the earlier discussion on characters of M_{p^∞}, the fact that ${}_{\mathfrak{c}}\zeta_p^F$ satisfies the (0) property implies that it suffices to check the interpolation property for the functions $\psi \cdot \mathbf{N}^k$ as ψ ranges across characters of of G_{p^∞} (included as any cell) and characters of G_1 (considered as functions on M_{p^∞} by inflation). Given a fixed $\mathfrak{b}$ and two indices i, j for which $I_i\mathfrak{b} = (\varpi)I_j$ as before, observe that we have a commutative diagram

$$ \begin{array}{ccc} (I_j)_p & \xrightarrow{\times\varpi} & (I_i)_p \\ \downarrow{\scriptstyle f_j} & & \downarrow{\scriptstyle f_i} \\ M_{p^\infty} & \xrightarrow{\times\mathfrak{b}} & M_{p^\infty} \end{array} \tag{4.31} $$

where we write, e.g., f_i for the restriction of f to the $[I_i]$-component. Thus, assembling (4.30) across ideal classes yields

$$ (\psi([\mathfrak{b}])\mathbf{N}\mathfrak{b}^k - 1){}_{\mathfrak{c}}\zeta_p^F(\psi \cdot \mathbf{N}^k) = (-1)^{n-1}(\psi([\mathfrak{b}])\mathbf{N}\mathfrak{b}^k - 1)(\zeta_F(\psi, -k) - \mathbf{N}c^{1+k} \cdot \zeta_F(\psi, -k)), \tag{4.32} $$

the right-hand side factorization following formally from factoring Dirichlet series, and the left-hand side factorization following from the fact that ψ is a character, allowing us to factor out a $\psi(\mathfrak{b})$ from the $\times\mathfrak{b}$-shifted terms coming from Proposition 4.22.

Then the only case in which we cannot find $\mathfrak{b}$ such that the Euler factor is nonzero is the subcase of (2) when $\psi = 1$ is the trivial character and $k = 0$; in all other cases, canceling the nonzero Euler factor yields the desired interpolation. But in this exceptional case, $\zeta_F(1, 0) = 0$ as well: indeed, (4.13) implies that the terms coming from pairing $[I_i]$ and $[I_i \cdot \mathfrak{a}]$ in the sum (4.26) cancel, for any fixed principal ideal $\mathfrak{a}$ generated by an element having negative norm. The result follows. □

As noted previously, Theorem 1.3 from the introduction is an immediate corollary.

4.4.3. *Extra 2-adic congruences of Deligne-Ribet.* As an application of our formalism, we recover the exceptional 2-adic congruences for totally odd functions proven in [DR, Theorems 8.11, 8.12]. We state here the theorem in the same maximum generality as loc. cit.; sadly, however, there are some delicate edge cases for which we have no cohomolgical account, and can only rely on the original automorphic arguments. We specify exactly what the missing piece is at the end of the proof below.

Theorem 4.23 (Theorem 1.4)**.** *Let $\psi : M_{2^\infty} \to \overline{\mathbb{Q}_2}$ be a totally odd continuous function of compact support. Then*

$$ \int_{M_{(2^\infty)}} \psi(J)\, d_{\mathfrak{c}}\zeta_2^F(J) \equiv 0 \pmod{2^{n-1}}. $$

Furthermore, we even have the stronger congruence

$$ \int_{M_{(2^\infty)}} \psi(J)\, d_{\mathfrak{c}}\zeta_2^F(J) \equiv 0 \pmod{2^n} $$

except *in the following case:*

- *the invariant*

$$ \delta(\psi) := \sum_{x \in G_1/\Sigma} \psi(x) \pmod 2 $$

equals $1 \in \mathbb{Z}/2$*, where here we view* $G_1 \hookrightarrow G_{2^\infty}$ *via the earlier-discussed inclusion,*[15] and

- *all units have positive norm and all combinations of signs with positive norm occur (*F *is* exceptional*, in the terminology of Deligne–Ribet),* and
- *the prime* $\mathfrak{c}$ *is inert in the extension* M/F *given by adjoining square roots of all totally positive units.*

Note in particular that the second condition means that the stronger congruence is satisfied for any totally odd character ψ *except possibly if it is totally unramified, i.e. factors through the narrow Hilbert class group.*

Proof. By continuity, we may assume ψ is locally constant; thus, we can represent the elements in Σ by actual principal ideals at some fixed level 2^k for some sufficiently large integer k, and we may consider integrals of totally odd functions ψ on $M_{(2^k)}$ against the image of ${}_{\mathfrak{c}}\zeta_2$ considered in $\mathbf{D}(M_{(2^k)}, \mathbb{Z}_2)$.

Note that if ψ is totally odd, then $\mathcal{D}^*\psi$ is totally odd on the "cell" $G_{(2^k)\mathfrak{d}^{-1}}$ for each k (noting that the cell decomposition is Σ-equivariant). Thus, it suffices to prove the theorem separately for odd functions supported on each of these open cells, together with functions inflated from the unramified quotient $M_{(2^\infty)} \to M_1 \cong G_1$, by a similar argument as before.

In the former case, note that we always want divisibility by 2^n, as remarked in the theorem statement. We consider $\mathcal{D}_*^{-1}{}_{\mathfrak{c}}\zeta_2 \in \mathbf{D}(G_{(2^k)\mathfrak{d}^{-1}}, \mathbb{Z}_2)$ for some $\mathfrak{d}|(2^k)$. The exponent k can be chosen to be any sufficiently large integer for ψ to be well-defined, so without loss of generality, we may consider k sufficiently large so that Σ acts freely on $G_{(2^k)\mathfrak{d}^{-1}}$. Then, by Proposition 4.15, for $\sigma \in \Sigma \cong \{\pm 1\}^{S_\infty}$, we have $[\sigma]_*{}_{\mathfrak{c}}\zeta_2 = \mathbf{N}\sigma \cdot {}_{\mathfrak{c}}\zeta_2$. This means ${}_{\mathfrak{c}}\zeta_2$ lies in the invariant submodule $(\mathbf{D}(G_{2^k\mathfrak{d}^{-1}}, \mathbb{Z}_2)(\mathbf{N}))^\Sigma$ (where we have written a norm twist on the Σ-action), since multiplication by a generator (in $G_{2^k\mathfrak{d}^{-1}}$) of σ is orientation preserving exactly when $\mathbf{N}\sigma = +1$ by definition; the same is then true for $\mathcal{D}_*^{-1}{}_{\mathfrak{c}}\zeta_2$ by Σ-equivariance of $\mathcal{D}$. We then have the exact sequence in Tate cohomology

$$\mathbf{D}(G_{2^k\mathfrak{d}^{-1}}, \mathbb{Z}_2)(\mathbf{N}) \xrightarrow{\mathrm{N}_\Sigma} (\mathbf{D}(G_{2^k\mathfrak{d}^{-1}}, \mathbb{Z}_2)(\mathbf{N}))^\Sigma \to H^0_T(\Sigma, \mathbf{D}(G_{2^k\mathfrak{d}^{-1}}, \mathbb{Z}_2)(\mathbf{N})) = 0.$$

Here,

$$\mathrm{N}_\Sigma = \sum_{\sigma \in \Sigma} \sigma$$

is the norm map for Σ, and the vanishing is by Shapiro's lemma for Tate cohomology, since Σ acts freely on $\mathbf{D}(G_{2^k\mathfrak{d}^{-1}}, \mathbb{Z}_2)(\mathbf{N})$. We thus have proven that the image of ${}_{\mathfrak{c}}\zeta_2$ in $\mathbf{D}(G_{2^k\mathfrak{d}^{-1}}, \mathbb{Z}_2)(\mathbf{N})$ is divisible by $\sum_{\sigma\in\Sigma}\sigma$, and [Gro, Lemma 5.3] implies that the evaluation against any totally odd ψ on $G_{2^k\mathfrak{d}^{-1}}$ is divisible by 2^n.

Finally, we consider the more delicate case, in which ψ is inflated from M_1. We fix a rank-$(n-1)$ subgroup $U^f \subset U$ containing U^+ such that $U \cong U^f \times \{\pm 1\}$; let $[U^f : U^+] = 2^s$. We fix a fundamental class c_{U^f} such that the corestriction of $\mathrm{cores}_{U^+}^{U^f} c_{U^+} = 2^s c_{U^f}$. Then for Σ_1 the image of Σ in G_1, we have $|\Sigma_1| = 2^{n-s-1}$.

If we now define ${}_{\mathfrak{c}}\theta_2^i$, for $1 \leqslant i \leqslant h$, to be the image of

$${}_{\mathfrak{c}}\Theta^{dR}(\alpha_i) \cap c_{U^f}^{\alpha_i}$$

in $\mathbf{D}(I_2^i/U^+, \mathbb{Z}_2)(\mathbf{N})_{U^f/U^+}$, and similarly

$${}_{\mathfrak{c}}\theta_2^F := \sum_{i=1}^h f_*{}_{\mathfrak{c}}\theta_2^i \in \mathbf{D}(M_{2^\infty}, \mathbb{Z}_2)(\mathbf{N})_{U^f/U^+},$$

[15] Note that the sum here is over the *wide* class group, and that this is well-defined for odd ψ only because of the reduction modulo 2.

then we can identify $2^s {}_{\mathfrak{c}}\theta_2^F$ with the image of ${}_{\mathfrak{c}}\zeta_2^F$ under the natural quotient

$$\mathbf{D}(M_{2^\infty}, \mathbb{Z}_2)(\mathbf{N}) \twoheadrightarrow \mathbf{D}(M_{2^\infty}, \mathbb{Z}_2)(\mathbf{N})_{Uf/U^+}.$$

Note that because ψ is unramified, it factors through it factors through the quotient to $\mathbf{D}(G_1, \mathbb{Z}_2)(\mathbf{N})$, which factors through the quotient above. Write ${}_{\mathfrak{c}}\theta[1]_2^F$ for the image of ${}_{\mathfrak{c}}\theta_2^F$ in $\mathbf{D}(G_1, \mathbb{Z}_2)(\mathbf{N})$; Proposition 4.15 implies it is Σ_1-invariant, as in the ramified case. Then also as before, we have the sequence in Tate cohomology

$$\mathbf{D}(G_1, \mathbb{Z}_2)(\mathbf{N}) \xrightarrow{\mathrm{N}_{\Sigma_1}} (\mathbf{D}(G_1, \mathbb{Z}_2)(\mathbf{N}))^{\Sigma_1} \to H_T^0(\Sigma_1, \mathbf{D}(G_1, \mathbb{Z}_2)(\mathbf{N})) = 0,$$

from which it follows (as in the ramified case) using [Gro, Lemma 5.3] that $|\Sigma_1| = 2^{n-s-1}$ divides ${}_{\mathfrak{c}}\theta[1]_2^F(\psi)$, and thus ${}_{\mathfrak{c}}\zeta_2^F(\psi)$ is always divisible by 2^{n-1}.

If F contains a unit of negative norm, then all the unramified totally odd L-values are zero, so improving this divisibility to 2^n becomes a trivial statement. To understand the extra divisibility when $\delta(\psi) = 0$ as in the theorem statement, then following [Gro], we can explicate the preceding cohomological calculation as follows: there then exists a lift (under N_Σ) of ${}_{\mathfrak{c}}\theta[1]_2^F$ to some element ${}_{\mathfrak{c}}\widetilde{\theta[1]}_2^F \in \mathbf{D}(G_1, \mathbb{Z}_2)$, such that we have a commutative diagram

$$\begin{array}{ccc} \mathbf{D}(G_1, \mathbb{Z}_2)(\mathbf{N}) & \xrightarrow{\psi} & \mathbb{Z}_2(\psi) \\ \big\downarrow{\scriptstyle N_\Sigma} & & \big\downarrow{\scriptstyle \times 2^{n-s-1}} \\ \mathbf{D}(G_1, \mathbb{Z}_2)(\mathbf{N}) & \xrightarrow{\psi} & \mathbb{Z}_2(\psi) \end{array} \tag{4.33}$$

for any totally odd ψ. Thus, to get an extra divisibility by 2 of ${}_{\mathfrak{c}}\theta[1]_2^F(\psi)$, it would suffice to show that ${}_{\mathfrak{c}}\widetilde{\theta[1]}_2^F(\psi)$ is divisible by 2.

But observe that reduced modulo 2, N_Σ actually factors through the quotient

$$\mathbf{D}(G_1, \mathbb{Z}/2)(\mathbf{N})_{\Sigma_1} \twoheadrightarrow \mathbf{D}(G_1/\Sigma_1, \mathbb{Z}/2).$$

In this quotient, the image of the reduction of ${}_{\mathfrak{c}}\widetilde{\theta[1]}_2^F$ modulo 2 is simply given by the formula for the map δ from the theorem statement. Then the condition that the evaluation of this at ψ exactly becomes $\delta(\psi) = 0$, as desired.

Observe now that the quantity

$$\frac{{}_{\mathfrak{c}}\zeta_2^F(\psi)}{2^{n-1}} \pmod 2$$

is actually independent of the choice of totally odd function ψ with $\delta(\psi) \equiv 1$, since the difference of two such functions has image under δ equal to zero. In this case, one can calculate directly from the Dirichlet series that as a function of $\mathfrak{c}$, the above expression actually gives a *homomorphism* $G_{2^\infty} \to \mathbb{Z}/2$ (see [DR, (8.11)]).[16]

Deligne–Ribet are able to prove that this homomorphism is nontrivial if and only if F is exceptional, and that in this case, it is the Artin reciprocity map for the Kummer extension M/F (as in the theorem statement). From this, the theorem's characterization of the remaining cases in which we have divisibility by 2^n in the unramified case follow. A purely cohomological account for these class field-theoretic interpretations eludes us, but we would be very interested to learn one. ☐

Appendix A. Comparison with a cyclotomic equivariant polylogarithm

In this appendix, working over the complex numbers, we argue that our de Rham *left* cohomology class

$$\Theta^{dR}(n) \in H^{n-1}(\mathrm{SL}_n(\mathbb{Z}), \mathcal{M}_{\mathbb{G}_m^n}/\langle d\log z_1 \wedge \ldots d\log z_n\rangle)$$

[16] Note that this would correspond to a stabilization identity similar to Proposition 4.22 in our setting, but since we already have proven the interpolation property, it also follows more simply from the identity of complex L-values.

coincides with the *right* cohomology class

$$S_{mult}[\{1\}] \in H^{n-1}(\mathrm{SL}_n(\mathbb{Z}), \mathcal{M}_{\mathbb{G}_m^n})$$

of [BCG2, Théorème 1.7], under the anti-involution $\gamma \mapsto \gamma^{-1}$ of $\mathrm{SL}_n(\mathbb{Z})$. This latter class is only well-defined up to the ambiguity $H^{n-1}(\mathrm{SL}_n(\mathbb{Z}), \mathbb{C})$, so what we really mean by this is that $S_{mult}[\{1\}]$ coincides with the $H^{n-1}(\mathrm{SL}_n(\mathbb{Z}), \mathbb{C})$-torsor of lifts of $\Theta^{dR}(n)$ defined from transgressions of the Euler class.

To briefly recall the definition of $S_{mult}[\{1\}]$, it is obtained from an equivariant ("polylogarithm") cohomology class

$$\mathcal{Z}_n \in H^{2n-1}_{\Gamma}(\mathbb{G}_m^n - \{1\}, \mathbb{C})^{(0)}$$

which is characterized (up to ambiguity $H^{2n-1}_{\Gamma}(\mathbb{G}_m^n, \mathbb{C})^{(0)}$) by having image a positive generator of $\{1\} \in H^{2n}_{\Gamma}(\{1\})$ under the residue map, along with its trace-invariance. Then if $H \subset \mathbb{G}_m$ is a suitable set of hyperplanes through zero, the image of $\mathcal{Z}_n$ under the edge map

$$H^{2n-1}_{\Gamma}(\mathbb{G}_m^n - H, \mathbb{C})^{(0)} \to H^{n-1}(\Gamma, H^n(\mathbb{G}_m^n - H)^{(0)})$$

together with a "formality" isomorphism

$$H^n(\mathbb{G}_m^n - H)^{(0)} \cong (\Omega^n_{\mathbb{G}_m^n - H})^{(0)}$$

yields $S_{mult}[0]$, up to the earlier-specified ambiguity.[17]

Let

$$\mathcal{D}^0 \to \ldots \to \ldots \mathcal{D}^{2n}$$

be the distributional de Rham complex computing the de Rham cohomology of $\mathbb{G}_m^n$; then the equivariant cohomology can be computed by the double complex of inhomogeneous cochains $C^\bullet(\mathrm{SL}_n(\mathbb{Z}), \mathcal{D}^\bullet)$; see [RX2] or the author's thesis [X].

Fix a Δ-extension E and an Euler transgression $\phi \in C^{n-1}(\mathrm{SL}_n(\mathbb{Z}), \mathbb{C})$; then one computes by the Poincaré-Lelong formula that the map

$$[\Delta^E(r_1, \ldots, r_k)] \mapsto \begin{pmatrix} r_1 & \ldots & r_k \end{pmatrix}_* (d\log)^{\wedge k}\{1 - z_1, \ldots, 1 - z_k\}$$

is a $\mathrm{SL}_n(\mathbb{Z})$-equivariant map $\widetilde{\mathrm{Chains}}(n)_\bullet \to \mathcal{D}^{2n-\bullet}$. Then the sum of the lifts from Lemma 2.5

$$\ell_1 + \ell_2 + \ldots + \ell_n - \phi \cdot (d\log)^{\wedge n}\{-z_1, \ldots, -z_k\}$$

is a trace-fixed class which has total coboundary $\{1\}$, and hence represents $\mathcal{Z}_n$. Its image under the Hochschild-Serre edge map is $(d\log)^{\wedge k}\theta_{E,\phi}(n)$, valued in trace-invariant holomorphic forms, and which therefore is identified with $S_{mult}[\{1\}]$ under the formality isomorphism up to the specified ambiguity.

By the same argument, using a stabilized equivariant class ${}_c\mathcal{Z}_n$, the class ${}_c\Theta^{dR}(n)$ coincides with $S_{mult}[\mu_c^n - \{1\}]$, where now the hyperplanes H can be taken to avoid lines through $\mu_{p^\infty}^n$ (and thus the resulting cohomology class extends holomorphically over all p-power torsion). (If one does this with c-smoothing rather than c-avoiding, one also can eliminate the ambiguity coming from the orientation obstruction.)

Appendix B. Duality in restricted Tits buildings

In this appendix, we prove that ${}^c_b\delta_*{}^c_b\Theta^{St}(n)$ and $(\star^*)^b_c\Theta^{St}(n)$ yield the same class in $H^{n-1}(\mathrm{GL}_n(\mathbb{Z}), (\star^*)^b_c\mathrm{St}(n))$. To give the idea of the argument, we first treat the unstabilized version: observe that the unstabilized duality map δ of Proposition 4.8 descends

[17] Using instead the approach of Kings-Sprang [KS] and starting with a polylogarithm class in *coherent* cohomology instead, the formality isomorphism can be avoided.

to a $\mathrm{GL}_n(\mathbb{Q})$-equivariant map

$$\delta : \mathrm{St}(n) \to (\star^*)\mathrm{St}(n), [\ell_1] \wedge \ldots \wedge [\ell_n] \to [\ell_1^\vee] \wedge \ldots \wedge [\ell_n^\vee]$$

by the same argument in Proposition 4.18. It is indeed also the case that[18]

$$\delta_* \Theta^{St}(n) = -(\star^*)\Theta^{St}(n) \in H^{n-1}(\mathrm{GL}_n(\mathbb{Q}), (\star^*)\mathrm{St}(n)). \tag{B.1}$$

To see this, let T_n be the *Tits building* for $\mathrm{GL}_n(\mathbb{Q})$: the $(n-2)$-dimensional simplicial complex whose i-simplices are proper flags of length i in $\mathbb{Q}^n$, i.e. chains of proper inclusions of proper nonzero subspaces of $\mathbb{Q}^n$

$$V_1 \subset \ldots \subset V_i$$

The Solomon-Tits theorem says that T_n has the homotopy type of a wedge of $(n-2)$-spheres. The group $\mathrm{GL}_n(\mathbb{Q})$ acts on T_n via its standard action on $\mathbb{Q}^n$, and the only nontrivial reduced homology group $\widetilde{H}_{n-2}(T_n)$ is then a model for the Steinberg representation $\mathrm{St}(n)$.

To describe the identification

$$\mathrm{St}(n) \xrightarrow{\sim} \widetilde{H}_{n-2}(T_n),$$

we note that given the standard set on $[n]$ elements $\{1, 2, \ldots, n\}$, we can form the standard $(n-1)$-simplex Δ^{n-1} whose i-faces correspond to the $(i+1)$-subsets of $[n]$, and inclusion of sets corresponds to the face relation. Then its first barycentric subdivision $\mathrm{sd}\,\Delta^{n-1}$ is also a contractible $(n-1)$-simplicial complex whose i-simplices are flags of non-empty subsets (i.e. chains of subsets of $[n]$ under proper inclusion).Then, as shown in [AR], the generator $[\ell_1] \wedge \ldots \wedge [\ell_n]$ (in our previous Orlik-Solomon-based notation) is identified with the image of the fundamental class (for the orientation corresponding to the usual order on $[n]$) under the simplex map

$$\partial \mathrm{sd}\,\Delta^{n-1} \hookrightarrow T_n \tag{B.2}$$

corresponding to the set $\ell_1, \ldots, \ell_n$, where we identify for example the flag of subsets $[1] \subset [1,3] \subset [1,3,4]$ with the flag of subspaces $\langle \ell_1 \rangle \subset \langle \ell_1, \ell_3 \rangle \subset \langle \ell_1, \ell_3, \ell_4 \rangle$.

Then the point is that using the lifting process in Lemma 2.5, one straightforwardly computes that the class $\theta^{St}(n)$ arises by lifting the $\mathrm{GL}_n(\mathbb{Q})$-fixed element $1 \in \mathbb{Z}$ along the map $H^0(\mathrm{GL}_n(\mathbb{Q}), \mathbb{Z}) \to H^{n-1}(\mathrm{GL}_n(\mathbb{Q}), \widetilde{H}_{n-2}(T_n))$ coming from the exact doubly-augmented homology complex

$$C(T_n)_\bullet := \widetilde{H}_{n-2}(T_n) \to C_{n-2}(T_n) \to \ldots \to C_0(T_n) \to \mathbb{Z}, \tag{B.3}$$

in the same way as we previously did with the Orlik-Solomon complex.

The Tits building admits an automorphism $\delta_T : T_n \to T_n$ sending a vertex corresponding to a subspace $V \subset \mathbb{Q}^n$ to its orthogonal complement $V^\perp$ under the standard inner product, and similarly on the simplices corresponding to flags (where $\perp$ reverses the inclusion chains). Further, one checks that this automorphism sends the top-degree homology class corresponding to the symbol $[\ell_1] \wedge \ldots \wedge [\ell_n]$ to $-[\ell_1^\vee] \wedge \ldots \wedge [\ell_n^\vee]$; hence, it induces the automorphism δ on $\mathrm{St}(n)$.

As with the duality map δ, the map δ_T is not $\mathrm{GL}_n(\mathbb{Q})$-equivariant, but it *is* $\mathrm{GL}_n(\mathbb{Q})$-equivariant for the $\star$-twisted action on the target, so induces an equivariant automorphism of homology complexes

$$(\delta_T)_* : C(T_n)_\bullet \xrightarrow{\sim} (\star^*)C(T_n)_\bullet$$

[18] See following footnote.

which evidently fixes the class of $1 \in \mathbb{Z}$, and hence the corresponding derived classes in top homology. From this, the duality relation (B.1) follows immediately.[19]

To attack the stabilized case, let $(b) = \mathfrak{b} \cap \mathbb{Z}$, $(c) = \mathfrak{c} \cap \mathbb{Z}$, and fix a c-torsion line λ and a b-torsion line μ such that neither is the reduction of the basis vector e_1; let $\Gamma \subset \mathrm{GL}_n(\mathbb{Z})$ be the congruence subgroup fixing $\lambda^\perp$ and μ, so that $\star\Gamma \subset \mathrm{GL}_n(\mathbb{Z})$ is the subgroup fixing λ and $\mu^\perp$. Then write ${}^{\lambda^\perp}_{\mu}\theta^{St}(n)$ for the Γ-cocycle

$$(\gamma_1, \ldots, \gamma_{n-1}) \mapsto [\gamma_1 \ldots \gamma_{n-1} e_1] \wedge \ldots \wedge [\gamma_1 e_1] \wedge [e_1] \in {}^{\lambda^\perp}_{\mu}\mathrm{St}(n)$$

ad ${}^{\lambda^\perp}_{\mu}\Theta^{St}(n)$ for its corresponding class in $H^{n-1}(\Gamma, {}^{\lambda^\perp}_{\mu}\mathrm{St}(n))$; switching the roles of c and b, we can analogously produce

$$[{}^{\mu^\perp}_{\lambda}\theta^{St}(n)] = {}^{\mu^\perp}_{\lambda}\Theta^{St}(n) \in H^{n-1}(\star\Gamma, {}^{\mu^\perp}_{\lambda}\mathrm{St}(n)).$$

Then Shapiro's lemma reduces Proposition 4.19 to proving that[20]

$$({}^{c}_{b}\delta)_{*m}{}^{\lambda^\perp}_{m} u\Theta^{St}(n) = -(\star^*){}^{\mu^\perp}_{\lambda}\Theta^{St}(n) \in H^{n-1}(\Gamma, (\star^*){}^{\mu^\perp}_{\lambda}\mathrm{St}(n)) \tag{B.4}$$

where we we are restricting ${}^{c}_{d}\delta$ to the summand ${}^{\lambda^\perp}_{\mu}\mathrm{St}(n)$, which it precisely maps to the summand ${}^{\mu^\perp}_{\lambda}\mathrm{St}(n)$.

Let ${}^{\lambda^\perp}_{\mu}T_n$ be the *restricted* Tits building consisting of the induced subcomplex of T_n on the vertices corresponding to subspaces which *do not contain* the line μ modulo d, and *are not contained* in $\lambda^\perp$ modulo c; let ${}^{\mu}_{\lambda}T_n$ be similar, but switching the role of λ and μ, c and b.

We write $C({}^{\lambda^\perp}_{\mu}T_n)_\bullet$ and $C({}^{\mu^\perp}_{\lambda}T_n)_\bullet$ for the respective doubly-augmented homology complexes of the form (B.3). Then the automorphism δ_T of T_n restricts to a map ${}^{\lambda^\perp}_{\mu}T_n \to {}^{\mu^\perp}_{\lambda}T_n$, and hence a Γ-equivariant map

$$(\delta_T)_* : C({}^{\lambda^\perp}_{\mu}T_n)_\bullet \to (\star^*)C({}^{\mu^\perp}_{\lambda}T_n)_\bullet.$$

It remains only to prove that ${}^{\lambda^\perp}_{\mu}T_n$ and ${}^{\mu^\perp}_{\lambda}T_n$ have reduced homology concentrated only in top degree, such that that we have an identification ${}^{\lambda^\perp}_{\mu}\mathrm{St}(n) \xrightarrow{\sim} \widetilde{H}_{n-2}({}^{\lambda^\perp}_{\mu}T_n)$ fitting in a Γ-equivariant commutative diagram

$$\begin{tikzcd} {}^{\lambda^\perp}_{\mu}\mathrm{St}(n) \arrow[r, "\sim"] \arrow[d, hook] & \widetilde{H}_{n-2}({}^{\lambda^\perp}_{\mu}T_n) \arrow[d] \\ \mathrm{St}(n) \arrow[r, "\sim"] & \widetilde{H}_{n-2}(T_n) \end{tikzcd} \tag{B.5}$$

and similarly for ${}^{\mu^\perp}_{\lambda}T_n$ and $\star\Gamma$. The conclusion would then follow by the same argument as the unstabilized case.

Since the two cases are symmetric in swapping λ and μ, c and d, we may as well focus only on the former case. Indeed, we note that the conditions defining ${}^{\lambda^\perp}_{\mu}\mathrm{St}(n)$ precisely state that it is generated by symbols

$$[\ell_1] \wedge \ldots \wedge [\ell_n]$$

[19]If we take de Rham realizations of $\mathrm{St}(n)$ directly using this duality, we obtain that $(\star^*)\Theta^{dR}(n)$ agrees with

$$(\gamma_1, \ldots, \gamma_{n-1}) \mapsto \begin{pmatrix} e & \gamma_1 e & \ldots & \gamma_1 \ldots \gamma_{n-1} e \end{pmatrix} \frac{(-1)^n z_1 \ldots z_n}{(1-z_1)\ldots(1-z_n)}$$

if taken to be valued in $\mathcal{M}_{\mathbb{G}_m^n}(-\det)^{(0)}$ modulo the $\mathrm{GL}_n(\mathbb{Q})$-orbit of classes of the form

$$(d\log)^{\wedge n}\{-z_1, \ldots, -z_i, 1 - z_{i+1}, \ldots, 1 - z_n\},$$

i.e. the image of all wedges (or, dually and equivalently, the Shintani generating functions of all lower-dimensional cones). The latter class is precisely the "naive Shintani cocycle" which [LP] proves is a cocycle modulo precisely these same relations.

[20]The appearance of the extra sign here, as well as in the unstabilized version (B.1), is explained by the fact that avoiding comes from lifting the sum over nontrivial torsion sections, smoothing comes from *negative* the sum over the nontrivial characters.

for which no line is contained in the $\lambda^\perp$ modulo c, and no $\leqslant (n-1)$-subtuple span contains μ modulo b. Thus, the simplex map (B.2) exists and makes sense when restricted to subspaces for the vertices of ${}^{\lambda^\perp}_{\mu}T_n$, satisfying all the Steinberg relations by the same arguments in [AR]. This definition for the top arrow evidently is Γ-equivariant and makes the diagram above commute, so the only thing that remains to prove is that it is an isomorphism.

Let us write $\mathrm{St}(n, \mathbb{F}_c)$, $\mathrm{St}(n, \mathbb{F}_b)$ for the Steinberg representation for lines modulo c, b respectively, and ${}^{\lambda^\perp}\mathrm{St}(n, \mathbb{F}_c)$, ${}_{\mu}\mathrm{St}(n, \mathbb{F}_b)$ for the same with the corresponding local conditions from the definition of ${}^{\lambda^\perp}_{\mu}\mathrm{St}(n)$.[21] Then by definition, we have a Cartesian square

(B.6)
$$\begin{array}{ccc} {}^{\lambda^\perp}_{\mu}\mathrm{St}(n) & \longrightarrow & \mathrm{St}(n) \\ \downarrow & & \downarrow \\ {}^{\lambda^\perp}\mathrm{St}(n, \mathbb{F}_c) \oplus {}_{\mu}\mathrm{St}(n, \mathbb{F}_b) & \longrightarrow & \mathrm{St}(n, \mathbb{F}_c) \oplus \mathrm{St}(n, \mathbb{F}_b) \end{array}$$

Likewise, let $T_n(\mathbb{F}_c), T_n(\mathbb{F}_b)$ be the Tits buildings built from subspaces of $\mathbb{F}_c^n$, $\mathbb{F}_b^n$, and ${}^{\lambda^\perp}T_n(\mathbb{F}_c), {}_{\mu}T_n(\mathbb{F}_b)$ the induced subcomplexes on subspaces not contained in $\lambda^\perp$, respectively not containing μ. Then we have a Cartesian square of simplicial complexes

(B.7)
$$\begin{array}{ccc} {}^{\lambda^\perp}_{\mu}T_n & \longrightarrow & T_n \\ \downarrow & & \downarrow \\ {}^{\lambda^\perp}T_n(\mathbb{F}_c) \oplus {}_{\mu}T_n(\mathbb{F}_b) & \longrightarrow & T_n(\mathbb{F}_c) \oplus T_n(\mathbb{F}_b) \end{array}$$

which leads to a Cartesian square also of the corresponding doubly-augmented homology complexes, and hence of the top-degree $\widetilde{H}_{n-2}$ groups, by the Künneth theorem. The left column of (B.6) is isomorphic to the $\widetilde{H}_{n-2}$ of the left column of (B.7) via the simplex maps (B.2); thus, if the same simplex map also induces isomorphisms

(B.8)
$${}^{\lambda^\perp}\mathrm{St}(n, \mathbb{F}_c) \xrightarrow{\sim} \widetilde{H}_{n-2}({}^{\lambda^\perp}T_n(\mathbb{F}_c)), {}_{\mu}\mathrm{St}(n, \mathbb{F}_b) \xrightarrow{\sim} \widetilde{H}_{n-2}({}_{\mu}T_n(\mathbb{F}_b))$$

then we also have our desired identification ${}^{\lambda^\perp}_{\mu}\mathrm{St}(n) \xrightarrow{\sim} \widetilde{H}_{n-2}({}^{\lambda^\perp}_{\mu}T_n)$. The maps (B.8) are symmetric to one another under the automorphism $V \mapsto V^\perp$ (and replacing c by b, etc.) so it suffices to prove the former. Indeed, this identification follows by the identification of the algebras $\mathscr{A}$ and $\mathscr{B}$ in [OS]: in particular, see the remark on p. 3 relating the Orlik-Solomon complex ($\mathscr{A}$) to the construction of Steinberg modules from simplicial complexes in [L] (agreeing with the construction of $\mathscr{B}$ in top degree); in the notation of loc. cit., our restricted Tits building is denoted $T(\mathbb{F}_c^n, \lambda^\perp)$, labeled case (b) on p. 551.

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[21] These are (up to $\perp$-duality) the top-degree parts of the Orlik-Solomon complex for what is called the rank-$(n-1)$ *affine geometry matroid* over $\mathbb{F}_c$ and $\mathbb{F}_b$, while the usual Steinberg module corresponds to the usual projective geometry matroid consisting of all lines.

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