LOCALIZATION AND MOTIVIC RELATIONS

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Abstract. We observe that the motivic analogue of Suslin reciprocity (and similar degree-zero statements) is a formal consequence of localization (plus purity/some six functor formalism). In particular, the statement of Suslin reciprocity for smooth schemes over fields due to Kriz is a corollary of localization for higher Chow groups, over any base; we write down the framework yielding such relations with coefficients for schemes smooth over any base in $\mathbb{A}^1$-invariant motivic cohomology. We refine some relations between cup products of modular units to be integral in coefficients and in the base as an example application.

1. Introduction and naive reciprocity

Let C be a smooth proper curve over a field k with function field K. The classical *Weil reciprocity* law states that the total degree of the divisor of a rational function, summing up multiplicities of poles and zeroes, is zero. This has the following generalization to Milnor K-theory: for each codimension-1 point x of C, write $\partial_x : K_n^M(K) \to K_{n-1}^M(k(x))$ for the associated tame symbol map in Milnor K-theory, which may be explicitly calculated by decomposing products into elements of the form

$$\{\pi, u_1, \ldots, u_{n-1}\} \mapsto \{\overline{u_1}, \ldots, \overline{u_{n-1}}\} \tag{1}$$

for a uniformizer π in the valuation ring at x and units $u_1, \ldots, u_n$; here, the bar denotes reduction to the residue field $k(x)$. Then *Suslin reciprocity* is the statement that the sum over all tame symbols on $K_n^M(K)$, transferred by finite pushforward down to k,

$$\varphi \mapsto \sum_{x \in C^{(1)}} \mathrm{N}_{k(x)/k} \partial_x \varphi, \tag{2}$$

(noting this sum only has finitely many nonzero terms for any given element) is the zero map. For $n = 1$, this recovers Weil reciprocity.

Milnor K-theory is the "simplest part" of the more general theory of motivic cohomology, a fundamental geometric/arithmetic invariant which may be defined for schemes more general than spectra of fields. More precisely, the bigraded motivic cohomology theory on schemes X (whose definition(s) we will examine later)

$$X \mapsto H^p(X, \mathbb{Z}(q))$$

coincides with the nth Milnor K-theory group in the regime $p = q = n$ for X the spectrum of a field. Sophie Kriz [Kr] formulated and proved the following generalization for the motivic cohomology of smooth schemes over a perfect field k.

Theorem 1.1. *Let S be a smooth variety over a perfect field k, $f : C \to S$ be smooth projective of relative dimension 1, and let $T \subset C$ be a reduced closed subscheme of codimension 1 whose* *components are all dominant over S. Then the composite*

$$H^p(C-T,\mathbb{Z}(q)) \xrightarrow{\partial} H^{p+1}_T(C,\mathbb{Z}(q-1)) \xrightarrow{(f|_T)_*} H^{p-1}(S,\mathbb{Z}(q-1))$$

is zero, where the two maps are given, respectively, by the localization sequence in higher Chow groups, and pushforward of cycles.

Kriz's proof interplays the Morel–Voevodsky presheaves-with-transfers model for triangulated mixed motives over a field with Bloch's classical theory of higher Chow groups. Our first observation is that, in fact, the existing theory of higher Chow groups alone affords a proof of Theorem 1.1 even over more general bases and with *no* smoothness hypothesis.

We quickly recall the higher Chow group theory: let X be *any* finite-type scheme over a regular base of dimension at most 1. Then after Bloch [Bl] (in the field case) and Levine [L2], the Zariski sheaves of higher cycle complexes on X are

$$\mathbb{Z}(n)^{BL}_X := \big(U \mapsto z^n(U\times\Delta^\bullet)\big)$$

where $z^n(U\times\Delta^i)$ is the module of codimension-n cycles on the algebraic i-simplex Δ^i over U which intersect all simplicial faces properly, with the natural face differentials. The higher Chow groups of X are then defined by

$$\mathrm{CH}^p(X,q) := \mathbb{H}^q(X, z^p(U\times\Delta^\bullet))$$

and inherit proper pushforward and flat pullback functoriality from the complexes. When X is defined over a field, the hypercohomology spectral sequence collapses immediately and the higher Chow groups may be computed simply from the complex of global sections $z^p(X\times\Delta^\bullet)$; in general, for the same reason, the higher Chow groups may always be computed in hypercohomology after pushforward to the base B rather than on all of X. Note that none of this requires any smoothness hypotheses, but the definition only generally gives the "correct" definition of motivic cohomology when X is smooth over B, in the sense of natural identifications $\mathrm{CH}^q(X,2q-p)\cong H^p(X,\mathbb{Z}(q))$.[1]

In this generality, Levine constructed a localization triangle [L2, Theorem 1.7]: for $i: Z\to X$ an equidimensional closed immersion of codimension d, with $j: U\to X$ be the complementary open immersion, and writing π for the projection map down to B of all three schemes, this is a distinguished triangle in the derived category

$$\pi_*\mathbb{Z}(n-d)^{BL}[-d] \xrightarrow{i_*} \pi_*\mathbb{Z}(n)^{BL}_X \xrightarrow{j^*} \pi_*\mathbb{Z}(n)^{BL}_U.$$

The localization sequence has as immediate corollary the following strengthening of Theorem 1.1, which comes from assuming f of relative dimension 1 and C,S smooth over a field in the statement below.

Theorem 1.2. *Let $f: C\to S$ be a proper map of schemes over B, and let $T\subset C$ be a closed subscheme of C of codimension d. The composition*

$$\mathrm{CH}^q(C-T,2q-p) \xrightarrow{\partial} \mathrm{CH}^{q-d}(T,2q-p) \xrightarrow{(f|_T)_*} \mathrm{CH}^{q-d}(S,2q-p)$$

[1] Morally, Bloch–Levine's higher Chow groups are the *Borel–Moore homology* theory for motivic spaces.

is zero, where the maps come from the connecting map in the localization sequence for $T \hookrightarrow C$, respectively the pushforward of the restriction of f.

Proof. The map $(f|_T)_*$ factors through the pushforward by the inclusion $T \hookrightarrow C$, so this follows from the composite of two consecutive arrows in the localization triangle being zero. □

In this optic, Suslin reciprocity, or more generally "sum of residues is zero," is a completely formal consequence of localization sequences. For applications, the interest often comes from the case when T is the disjoint union of some set of components $(T_i)_{i\in I}$, in which case we obtain relations analogous to (2) in motivic cohomology coming from the vanishing of sums of operators

$$\sum_{i\in I}(f|_{T_i})_* \circ \partial_{T_i} = 0,$$

whose *individual* terms can be interesting non-vanishing classes in the motivic cohomology of S. For instance, this is the use case in our joint work establishing certain relations among special motivic elements on Drinfeld modules [SX], whose number field analogues (one of which is treated using Milnor K-theory of the generic point in [BPPS]) motivated us to write this note.

In the remainder of this short note, we record a slightly more sophisticated formulation of the above observation in the setting of $\mathbb{A}^1$-invariant motivic cohomology with coefficients (Theorem 2.3), and refine the main results of [BPPS] to motivic cohomology groups integral in the coefficients and base (Theorem 3.4).

1.1. **Acknowledgements.** Thanks to Sophie Kriz for responding to emails and expressing interest, Cecilia Busuioc for a motivating conversation on applications, and Tess Bouis for some helpful guidance on motives.

2. Localization in SH and motivic relations

In this section, we record (Theorem 2.3) that the literature contains the needed formalism to obtain a general "Suslin reciprocity" type of statement in the setting of $\mathbb{A}^1$-invariant motivic cohomology with coefficients, and then use Spitzweck's model to check that the maps involved coincide with the "naive" ones in the introduction in the absolutely smooth case (i.e. the regime in which higher Chow groups are expected to coincide with motivic cohomology). We note that for the example application in the following section, we actually only need the trivial-coefficients setting, but we believe the more general formulation may be useful in understanding relations between motivic polylogarithms.

Let X be a separated Noetherian scheme over B. We write $\mathrm{Sm}_{X,t}$ for the category of smooth X-schemes endowed with a Grothendieck topology t, and $\mathrm{Sh}(\mathrm{Sm}_{X,t}, \mathrm{Ani})$ the ∞-category of sheaves of anima on the associated site. Let $\mathbb{T}_X := \mathbb{A}^1/\mathbb{G}_m$ be the Tate object in $\mathrm{Sh}(\mathrm{Sm}_{X,Zar}, \mathrm{Spc})$, defined as the cofiber of the indicated inclusion of represented sheaves.

Definition 2.1. We have the following categories associated to X:

- The *unstable motivic homotopy category* $\mathrm{H}(X)$ is the localization of the homotopy category of $\mathrm{Sh}(\mathrm{Sm}_{X,Nis}, \mathrm{Spc})$ along all projection maps $S \times \mathbb{A}^1 \to S$ (or rather the associated sheaves of sets) for any $S \in \mathrm{Sm}_{X,Nis}$.
- The *stable homotopy category* $\mathrm{SH}(X) := \mathrm{H}_*(X)[\mathbb{T}_X^{-1}]$ is obtained by freely smash-inverting the Tate object in the *pointed* motivic homotopy category (i.e. the under-category of the terminal object).

One defines an oriented E_∞-ring spectrum $H\mathbb{Z}_X$ in the stable homotopy category $\mathrm{SH}(X)$ via the slice filtration on $K\mathrm{GL}$ [BEM], and one then defines the triangulated category of motives over X

$$\mathrm{DM}(X) := \mathrm{Ho}(H\mathbb{Z}_X\text{-Mod})$$

where the monoidal structure is the smash product of $H\mathbb{Z}_X$-modules, and sets

$$H^i(X, \mathbb{Z}(n)) := \mathrm{Hom}_{\mathrm{SH}(X)}(1, \Sigma^{i,n} H\mathbb{Z}_X) \cong \mathrm{Hom}_{\mathrm{DM}(X)}(\mathbb{Z}(0)_X, \mathbb{Z}(n)_X[i]),$$

the latter isomorphism being the free/forgetful adjunction between the free $H\mathbb{Z}_X$-module functor $\mathrm{SH}(X) \to \mathrm{DM}(X), \mathcal{S} \mapsto H\mathbb{Z}_X \wedge \mathcal{S}$ and the forgetful inclusion functor $\mathrm{DM}(X) \to \mathrm{SH}(X)$. Here $\Sigma^{i,n}$ denotes the bigraded suspension functor in $\mathrm{SH}(X)$ coming from smash product with $(S^1)^{\wedge i} \wedge \mathbb{T}^{\wedge n}$, and $\mathbb{Z}(k)_X$ is the Tate twisted coefficient module

$$\mathbb{Z}(k)_X := H\mathbb{Z}_X \wedge \mathbb{T}^{\wedge k}$$

where $H\mathbb{Z}_X$ acts on the left smash factor, so $\Sigma^{i,n} H\mathbb{Z}_X$ and $\mathbb{Z}(n)_X[i]$ are just two names for the same object (identified via the forgetful functor) in our chosen nomenclature in their respective categories. More generally, given $E \in \mathrm{DM}(X)$, we may set

$$H^i(X, E(n)) := \mathrm{Hom}_{\mathrm{SH}(X)}(1, \Sigma^{i,n} E) \cong \mathrm{Hom}_{\mathrm{DM}(X)}(\mathbb{Z}(0)_X, E(n)[i]),$$

The base-change compatibility property $f^* H\mathbb{Z}_Y \cong H\mathbb{Z}_X$ for $f : X \to Y$ implies that the motivic cohomology of X may always equally be computed on the base as $H^i(Y, f_* E(n))$, as one expects.

We appeal to [Hoy] for the six-functor formalism in $\mathrm{SH}(X)$: for a separated morphism of finite type $f : X \to Y$, there are adjoint pairs of functors

$$f^* : \mathrm{SH}(Y) \rightleftarrows \mathrm{SH}(X) : f_*, \quad f_! : \mathrm{SH}(X) \rightleftarrows \mathrm{SH}(Y) : f^!$$

satisfying a series of compatibilities, including that $f_* = f_!$ for proper f. The following are our needed formal properties:

Proposition 2.2. *Let S be a qcqs scheme, and write*

$$X \longmapsto \mathrm{SH}(X)$$

for the stable motivic homotopy category. The following properties hold (e.g. as recorded in [Hoy, Theorem 6.18], or [MV, Theorem 3.2.23] for purity of a closed pair).

(1) (localization) *Let $i : Z \hookrightarrow X$ be a closed immersion with open complement $j : U \hookrightarrow X$. Then for every $E \in \mathrm{SH}(X)$ there is a functorial localization triangle*

$$i_* i^! E \longrightarrow E \longrightarrow j_* j^* E \xrightarrow{\partial_Z} i_* i^! E[1].$$

(2) (relative purity) *If $f : X \to Y$ is smooth, then there is a functorial purity equivalence $f^! E \simeq f^* E \wedge \mathrm{Th}(T_f)$ where T_f is the relative tangent bundle.*

(3) (purity of a smooth closed pair) *If $i : Z \hookrightarrow X$ is a closed immersion of smooth schemes over a base S, let $N_{Z/X}$ be the normal bundle. Then there is a functorial purity equivalence*

$$i^! E \simeq i^* E \wedge \mathrm{Th}(-N_{Z/X}).$$

The same formal properties are then inherited for the categories of modules over the base-change compatible assignment of oriented E_∞-ring spectra $S \mapsto \mathrm{DM}(S)$. In this setting, for f smooth of relative dimension d, relative purity becomes

$$f^! E \simeq f^* E(d)[2d].$$

and if i has codimension c, purity of a smooth closed pair becomes

$$i^! E \simeq i^* E(-c)[-2c]$$

by identifying the respective Thom spectra with powers of the Tate sphere. From these formal properties, it follows in much the same way as usual:

Theorem 2.3. *Let $f : C \to S$ be proper of relative dimension d, and $i : T \hookrightarrow C$ be a codimension-d closed immersion of schemes smooth over S with open complement $j : C - T \hookrightarrow C$. Write $g = f \circ i$. Let $E \in \mathrm{DM}(S)$ be any $H\mathbb{Z}_S$-module spectrum. Then the composite in $\mathbb{A}^1$-invariant motivic cohomology*

$$H^p(U, j_* f^* E(q)) \to H^{p-2d+1}(T, g^* E(q-d)) \to H^{p-2d+1}(S, E(q-d))$$

is the zero map, where the two maps coming from the localization sequence, respectively the trace map for the finite morphism $g : T \to S$ via the composite in $\mathrm{DM}(S)$

$$g_* g^* E(q-d) = g_! g^! E(q-d) \to E(q-d)$$

coming from the counit of the adjunction, where here we have used relative dimension-0 purity and the identification $g_* = g_!$ for g proper.

Proof. From localization and the two purities, we have a triangle in $\mathrm{DM}(C)$

$$i_* g^* E(d)[2d] \to f^* E \to j_* j^* f^* E \to i_* g^* E(d)[2d+1]$$

to which we may apply $\mathrm{Hom}_{\mathrm{DM}(X)}(\mathbb{Z}(0)_X, - \otimes \mathbb{Z}(q)_X)$ to obtain the standard localization sequence with supports, including the segment

$$\ldots \to H^p(U, j_* f^* E(q)) \to H^{p-2d+1}(T, g^* E(q-d)) \to H^{p+1}(C, E(q)) \to \ldots$$

Writing the closed immersion map on motivic cohomology from the localization triangle on S as

$$g_* g^* E(d)[2d] \to f_* f^* E$$

we see that the trace map for g factors through this morphism via the counit for the proper morphism f (with purity twist, as in the theorem statement but in dimension d rather than 0), so we are done as before. □

As mentioned, when $E = H\mathbb{Z}_S$, and all schemes are absolutely smooth, this construction agrees with the Bloch–Levine construction from the introduction: Spitzweck constructs [Sp, §4.4] an oriented E_∞-ring spectrum for X smooth over a Dedekind domain

$$M\mathbb{Z}_X \in \mathrm{Sp}_{\mathbb{T}}(\mathrm{Ch}(\mathrm{Sh}(\mathrm{Sm}_{X,Zar}, \mathbb{Z}))),$$

for which one has an isomorphism of oriented E_∞-ring spectra $M\mathbb{Z}_X \cong H\mathbb{Z}_X$ [BH] compatible with base change. For the purposes of calculation, Spitzweck constructs also a "naive spectrum" $\mathcal{M}_X \in \mathrm{SH}(X)$ and a functorial isomorphism

$$\mu_X : \mathcal{M}_X \xrightarrow{\sim} M\mathbb{Z}_X.$$

The spectrum $\mathcal{M}_X$, not being defined via a rectified symmetric $\mathbb{T}$-spectrum, cannot see the E_∞ structure, but has the advantage of being an $\Omega_{\mathbb{T}}$-spectrum in the sense that for all r, there are canonical isomorphisms

$$M_X(r-1)[-1] \cong \underline{\mathrm{Hom}}(\mathbb{T}_X, M_X(r))$$

inside $\mathrm{SH}(X)$, where on the right-hand side we have the mapping spectrum [Sp, Proposition 5.29], and $M_X(r)[r] \in D(\mathrm{Sh}(\mathrm{Sm}_{X,Zar}, \mathbb{Z}))$ is the "level r" part of $\mathcal{M}_X$ (i.e. the space E_r in the tower-of-spaces description of spectra).[2][3] This allows Spitzweck to calculate the comparison:

Proposition 2.4. *Let X be smooth over a Dedekind domain B, writing $f : X \to \operatorname{Spec} B$ for the structure map. The comparison μ gives rise to functorial isomorphisms*

$$\mathrm{Hom}_{\mathrm{SH}(X)}(1, \Sigma^{i,n} M\mathbb{Z}_X) \cong \mathrm{CH}^n(X, 2n-i)$$

Proof. This is left as an exercise by Spitzweck, so we give a few details. If $f : X \to Y$ is a smooth map, then the pullback of sheaves induces a functor $f^* : \mathrm{SH}(Y) \to \mathrm{SH}(X)$, which has a *left* adjoint $f_\# : \mathrm{SH}(X) \to \mathrm{SH}(Y)$ defined on the level of sheaves by left Kan extension along the obvious forgetful functor $\mathrm{Sm}_X \to \mathrm{Sm}_Y$. It follows then almost tautologically that the image of the unit $f_\# 1_{\mathrm{SH}(X)}$ may be identified with the suspension spectrum $\Sigma^\infty X_+ \in \mathrm{SH}(Y)$: i.e., the spectrum given in level n by $\mathbb{Z}[X]_{Zar} \wedge \mathbb{T}_Y^{\wedge n}$ with the tautological structure maps. Then we may compute:

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{SH}(X)}(1, \Sigma^{i,n} M\mathbb{Z}_X) &\cong \mathrm{Hom}_{\mathrm{SH}(Y)}(\Sigma^\infty X, \Sigma^{i,n} M\mathbb{Z}_Y) && (3)\\
&\cong \mathrm{Hom}_{\mathrm{SH}(Y)}(\Sigma^\infty X, \Sigma^{i,n} \mathcal{M}_Y(n)[i]) && (4)\\
&\cong \mathrm{Hom}_{D(\mathrm{Sh}(\mathrm{Sm}_{Y,Zar},\mathbb{Z}))}(\mathbb{Z}[X]_{Zar}, M_Y(n)[i]) && (5)\\
&\cong \mathbb{H}^i_{Nis}(X, M_Y(n)|_{X_{Nis}}) && (6)\\
&\cong \mathbb{H}^i_{Zar}(X, \mathbb{Z}(n)^{BL}_X) && (7)
\end{aligned}$$

which is the higher Chow group. Here, the identification (3) is the adjunction between $f_\#$ and f^* together with base change of $M\mathbb{Z}$, (4) is the identification μ_X, (5) is the $\Omega_{\mathbb{T}}$-spectrum property of $\mathcal{M}$, (6) is the derived Yoneda lemma, and (7) is [Sp, Corollary 5.20] together with

[2] Spitzweck does not distinguish notation for the spectra $\mathcal{M}_X$ and the level parts we denote M_X; we find it helpful for our understanding to do so.

[3] Here, one applies the Dold–Kan functor and Nisnevich sheafifies to view these objects inside $\mathrm{SH}(X)$; Spitzweck frequently implicitly views complexes of big site Zariski sheaves inside the category of motivic spaces in this way and we follow his lead.

the calculation of Geisser [G, Proposition 3.6] that Nisnevich and Zariski hypercohomologies of the Bloch cycle complexes compute the same groups. □

Spitzweck records ([Sp, Theorem 3.1]) that Levine's localization sequences assemble into a localization sequence for the spectra $\mathcal{M}_X$, whose maps one may check essentially tautologically are identified under μ with the localization sequence for $M\mathbb{Z}_X$.

3. Bykovskii relations among modular units

The main result of [BPPS] yields certain relations between cup products in Milnor K-theory of modular units, following an idea of Goncharov.[4] These results were stated with $\mathbb{Q}$-coefficients and "at the generic point" on complex-analytic models of elliptic curves; for arithmetic applications, e.g. to the Sharifi conjectures (cf. [SV], [LSSW]), one prefers integrality both in the coefficients and the base. Using the machine for producing motivic relations discussed in the previous sections, we prove such a refinement. We will give this topic a more systematic treatment using cycle-theoretic methods in forthcoming work; here, we restrict our attention to understanding integrality in the results of [BPPS] of elliptic schemes over regular bases and untwisted motivic coefficients. In particular, we here do not make real use of the more sophisticated formalism of the previous section, though we believe it may have interesting applications to relations between more general families of polylogarithm-like elements in the future.

Let $\pi : E \to S$ be an elliptic scheme over a regular base, and let $N > 2$ be an integer such that $E[N]$ is a constant group scheme over S; write $U = E - E[N]$ for the complement. (We treat here only the case where S is regular, but everything is the same in the general case except that all unit groups should be replaced with the groups of units on semi-normalizations as a consequence of cdh-sheafification from the formalism of the previous section).

The following is an algebraic argument substituting for the complex-analytic construction of the theta elements ${}_N\Theta_{\mathbf{a}}(u,\tau)^{12}/{}_N\Theta_{\mathbf{x}}(u,\tau)^{12}$ in loc. cit.:

Proposition 3.1. *Let D be a degree-zero divisor supported on $E[N]$, disjoint from zero. Write*

$$d_N := \left(\gcd_{a \equiv 1 \pmod{N}} a^2 - 1 \right)$$

Then there is a distinguished element $\theta_D \in H^1(U, \mathbb{Z}(1)) = \mathcal{O}(U)^\times$ with divisor $d_N \cdot N \cdot D$ characterized by its invariance under $[a]_*$ for every integer $a \equiv 1 \pmod{N}$.

Proof. The unit-divisor exact sequence (or the localization sequence more generally) gives

$$\mathcal{O}(S)^\times \cong \mathcal{O}(E)^\times \to \mathcal{O}(E - E[N])^\times \to \mathbb{Z}\{E[N]\} \to \mathrm{Pic}(E)(S)$$

The image of the class of D in the latter group is N-torsion after passing to its image in the *relative* Picard group $\mathrm{Pic}_S(E)(S)$ since the cycle class map identifies its identity component with E. Thus, $\mathcal{O}_E(ND)$ is the pullback of a bundle on S: Writing $e : S \to E$ for the identity section, we have $e^*\mathcal{O}_E(P) \cong \mathcal{O}_S$, so we conclude that the image of $\mathrm{lcm}(12, N)D$ is zero in

[4] This followed earlier work of Brunault [Bru] showing such relations at the level of étale cohomology; related results were also proven in [FK] and [SV].

$\mathrm{Pic}(E)(S)$, and that N alone is sufficient if D is disjoint from zero. To kill the lift ambiguity in $\mathcal{O}(S)^\times$, we observe that each projector $[a]_* - a^2$ for integers $a \equiv 1 \pmod N$ annihilates this group but acts on D by the scalar $1-a^2$. The least common multiple of these numbers as a varies is the other term in d_N. □

The uniqueness statement gives standard compatibilities between the various θ_D like multiplicativity in D, transformation under isogenies, etc.

Remark 3.2. The disjoint-from-zero bound $d_N \cdot N$ is always an improvement on the $12N^2$ of [BPPS]: one may compute that it is always either N^2, $2N^2$, or $4N^2$. We remark that if D is not disjoint from zero, $e^*\mathcal{O}_E(0) \simeq \omega^{-1}$ for ω the Hodge bundle on S; the 12th power of the Hodge bundle is famously trivial on the moduli stack of elliptic curves, so we need to multiply by $\mathrm{lcm}(12, N)$ rather than just N. Then for these theta functions, the overlapping-zero denominator may sometimes be $24N^2$, so this is not always an improvement.

For nonzero $\mathbf{u}, \mathbf{v} \in E[N](S)$, define the Siegel unit

$$G_{\mathbf{u},\mathbf{v}} := e^*\theta_{\mathbf{v}-\mathbf{u}} \in \mathcal{O}(S)^\times.$$

Let us write, for $\mathfrak{a}$ nonzero,

$$\mathfrak{g}_{\mathbf{a}} \in \mathcal{O}(S)^\times$$

for the classical integral Siegel unit normalized so that, over the complex modular curve, it is $g_{\mathbf{a}}^{d_N N}$ (cf. the formula written as [BPPS, Definition 1.1.1]). Then one may compute $G_{\mathbf{u},\mathbf{v}} = \left(\frac{\mathfrak{g}_{\mathbf{u}}}{\mathfrak{g}_{\mathbf{v}}}\right)^N$.

Now, if $\mathbf{u}, \mathbf{v}, \mathbf{w} \in E[N](S) - \{0\}$, choose nonzero $\mathbf{r} \in E[N](S)$ distinct from them and set

$$\kappa(\mathbf{u}, \mathbf{v}, \mathbf{w}) := \{G_{\mathbf{u},\mathbf{r}}, G_{\mathbf{v},\mathbf{r}}\} + \{G_{\mathbf{v},\mathbf{r}}, G_{\mathbf{w},\mathbf{r}}\} + \{G_{\mathbf{w},\mathbf{r}}, G_{\mathbf{u},\mathbf{r}}\} \in H^2(S, \mathbb{Z}(2)).$$

Since one may easily compute that $G_{\mathbf{a},\mathbf{b}}G_{\mathbf{b},\mathbf{c}} = G_{\mathbf{a},\mathbf{c}}$ from the multiplicativity properties of the theta function, this is actually independent of $\mathbf{r}$. Now, for pairwise distinct $\mathbf{a}, \mathbf{b}, \mathbf{c} \in E[N](S)$, define

$$\Psi(\mathbf{a} : \mathbf{b} : \mathbf{c}) := \kappa(\mathbf{a} - \mathbf{b}, \mathbf{c} - \mathbf{a}, \mathbf{b} - \mathbf{c}),$$

and declare it to be zero by fiat otherwise.

Lemma 3.3. *For any* $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{x} \in E[N](S)$*, we have*

$$N d_N (\Psi(\mathbf{a} : \mathbf{b} : \mathbf{c}) - \Psi(\mathbf{a} : \mathbf{b} : \mathbf{x}) - \Psi(\mathbf{a} : \mathbf{x} : \mathbf{c}) - \Psi(\mathbf{x} : \mathbf{b} : \mathbf{c})) = 0$$

(where this is essentially by fiat when the four sections are not pairwise distinct).

Proof. For $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{x}$ all nonzero, let

$$\Xi_{\mathbf{x}}(\mathbf{a}, \mathbf{b}, \mathbf{c}) := \{\theta_{\mathbf{a},\mathbf{x}}, \theta_{\mathbf{b},\mathbf{x}}, \theta_{\mathbf{c},\mathbf{x}}\} \in H^3(U, \mathbb{Z}(3)).$$

Then we may compute, exactly as in [BPPS, Lemma 3.2.1],

$$(\pi|_{E[N]})_* \partial \Xi_{\mathbf{x}}(\mathbf{a}, \mathbf{b}, \mathbf{c}) = N d_N (\Psi(\mathbf{a} : \mathbf{b} : \mathbf{c}) - \Psi(\mathbf{a} : \mathbf{b} : \mathbf{x}) - \Psi(\mathbf{a} : \mathbf{x} : \mathbf{c}) - \Psi(\mathbf{x} : \mathbf{b} : \mathbf{c}))$$

and then result then follows by Theorem 2.3. In general, notice that the identity only involves "homogeneous" terms unaffected by the translation $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{x} \mapsto \mathbf{a} + \mathbf{t}, \mathbf{b} + \mathbf{t}, \mathbf{c} + \mathbf{t}, \mathbf{x} + \mathbf{t}$, so we may simply pick such a $\mathbf{t}$ and apply the same argument for the translated sections. □

Then by Goncharov's finite-group trick applied to $\bigwedge^3 \mathbb{Z}[E[N](S)]$, exactly as in [BPPS, Proposition 3.2.3], we deduce that for *any* fixed $\mathbf{a}, \mathbf{b}, \mathbf{c} \in E[N](S)$, we have

$$2 \sum_{\mathbf{x} \in A} \Psi(\mathbf{a} : \mathbf{b} : \mathbf{x}) = 2 \sum_{\mathbf{x} \in A} \Psi(\mathbf{a} : \mathbf{x} : \mathbf{c}) = 2 \sum_{\mathbf{x} \in A} \Psi(\mathbf{x} : \mathbf{b} : \mathbf{c}) = 0.$$

from which, summing the preceding lemma over $\mathbf{x} \in E[N](S)$ as in loc. cit., we deduce:

Theorem 3.4. *Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in E[N](S)$ be pairwise distinct. Then*

$$2d_N N^4 \, \Psi(\mathbf{a} : \mathbf{b} : \mathbf{c}) = 0 \in H^2(S, \mathbb{Z}(2)).$$

Equivalently, if $\mathbf{u}, \mathbf{v}, \mathbf{w} \in E[N](S)$ satisfy $\mathbf{u} + \mathbf{v} + \mathbf{w} = 0$, then

$$2d_N N^4 \, \kappa(\mathbf{u}, \mathbf{v}, \mathbf{w}) = 0 \qquad \text{in } H^2(S, \mathbb{Z}(2)).$$

We have

$$\kappa(\mathbf{u}, \mathbf{v}, \mathbf{w}) = N^2(\{\mathfrak{g}_{\mathbf{u}}, \mathfrak{g}_{\mathbf{v}}\} + \{\mathfrak{g}_{\mathbf{v}}, \mathfrak{g}_{\mathbf{w}}\} + \{\mathfrak{g}_{\mathbf{w}}, \mathfrak{g}_{\mathbf{u}}\}),$$

so we can also rephrase this in the more familiar form:

Corollary 3.5. *Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in E[N](S) - \{0\}$ satisfy $\mathbf{u} + \mathbf{v} + \mathbf{w} = 0$. Then*

$$2d_N N^6 \, (\{\mathfrak{g}_{\mathbf{u}}, \mathfrak{g}_{\mathbf{v}}\} + \{\mathfrak{g}_{\mathbf{v}}, \mathfrak{g}_{\mathbf{w}}\} + \{\mathfrak{g}_{\mathbf{w}}, \mathfrak{g}_{\mathbf{u}}\}) = 0 \in H^2(S, \mathbb{Z}(2)).$$

As in loc. cit., this leads to modular symbols. Let $\mathrm{St}(\mathbb{Q}^n)$ be the usual Steinberg module of $\mathbb{Q}^n$ coming from the top reduced homology of the Tits building of $\mathrm{GL}_n(\mathbb{Q})$; it is generated by apartment classes $[v_1, \ldots, v_n]$ coming from unimodular bases of $\mathbb{Z}^n$. Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{Q}^n$.

Theorem 3.6. *There is a $\mathrm{GL}_n(\mathbb{Z})$-equivariant map*

$$\mathrm{St}(\mathbb{Q}^n) \to \mathrm{Hom}_{\mathrm{Set}}(\mathrm{Hom}(\mathbb{Z}^n, E[N](S)), H^n(S, \mathbb{Z}(n)))$$

sending

$$[v_1, \ldots, v_n] \mapsto (\phi \mapsto M\{\mathfrak{g}_{\phi(v_1)}, \ldots, \mathfrak{g}_{\phi(v_n)}\})$$

where we take the convention $\mathfrak{g}_0 = 1$ and $M = 2d_N N^6$. Here, the $\mathrm{GL}_n(Z)$-action on the target is $\mu \mapsto (\phi \mapsto \mu(\phi \circ \gamma))$ for the usual left action of $\gamma \in \mathrm{GL}_n(\mathbb{Z})$ on $\mathbb{Z}^n$.

Proof. The equivariance is formal assuming well-definedness. Using the unimodular Bykovskii presentation [Byk] of the Steinberg module $\mathrm{St}(\mathbb{Q}^n)$, it suffices to check that the image under the above-defined map of

$$[e_1, \ldots, e_n] - [e_1 + e_2, e_2, \ldots, e_n] + [e_1 + e_2, e_1, e_3, \ldots, e_n]$$

vanishes. This amounts to relations of the form

$$M(\{\mathfrak{g}_{\mathbf{x}_1}, \mathfrak{g}_{\mathbf{x}_2}\} - \{\mathfrak{g}_{\mathbf{x}_1+\mathbf{x}_2}, \mathfrak{g}_{\mathbf{x}_2}\} + \{\mathfrak{g}_{\mathbf{x}_1+\mathbf{x}_2}, \mathfrak{g}_{\mathbf{x}_1}\}) \smile \{\mathfrak{g}_{\mathbf{x}_3}, \ldots, \mathfrak{g}_{\mathbf{x}_n}\}.$$

Note that $\mathfrak{g}_{-\mathbf{x}} = \mathfrak{g}_{\mathbf{x}}$ for any $\mathbf{x}$. When any of $\mathbf{x}_1, \ldots, \mathbf{x}_n, \mathbf{x}_1 + \mathbf{x}_2$ are zero, then the relation degenerates into a triviality since the Steinberg symbol is alternating after multiplication by 2. Otherwise,

$$M(\{\mathfrak{g}_{\mathbf{x}_1}, \mathfrak{g}_{\mathbf{x}_2}\} - \{\mathfrak{g}_{\mathbf{x}_1+\mathbf{x}_2}, \mathfrak{g}_{\mathbf{x}_2}\} + \{\mathfrak{g}_{\mathbf{x}_1+\mathbf{x}_2}, \mathfrak{g}_{\mathbf{x}_1}\}) = M(\{\mathfrak{g}_{\mathbf{x}_1}, \mathfrak{g}_{\mathbf{x}_2}\} + \{\mathfrak{g}_{\mathbf{x}_2}, \mathfrak{g}_{-\mathbf{x}_1-\mathbf{x}_2}\} + \{\mathfrak{g}_{-\mathbf{x}_1-\mathbf{x}_2}, \mathfrak{g}_{\mathbf{x}_1}\})$$

is precisely a Manin relation of the form we established. ∎

Remark 3.7. As we have phrased it, everything done here only requires the naive higher Chow group formulation of section 1, besides the parenthetical remark at the beginning that all results hold even for S irregular by passing to the seminormalization as a result of the cdh-sheafified $\mathbb{Z}(1)^{\mathbb{A}^1}$ in [BEM]. We note also though the foundations in [Hoy] we used in section 2 are also valid for the modular stack, and so one can equally prove the above Bykovskii relations for elements in the motivic cohomology of that stack taking $N = 3$ (the case $N = 2$ runs into problems with the general position of $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{x}$ argument we used). This would also become relevant if we wanted relations at coarser than full level N, which also should exist integrally.

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