POLYLOGARITHM AND CYCLOTOMIC ELEMENTS

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We will show that a version of Deligne's story [D] gives a remarkably simple and coherent construction of cyclotomic elements in higher (rational) K–groups of cyclotomic fields; it also yields a proof of conjecture ( ) [BK] thus filling the gap in proof of Kato's theorem ( ) [BK] on the values of Riemann ζ–function.

The paper starts with a short review of Deligne's fundamental paper [D] (with the most advanced results, such as the crystalline games and precise torsion computations, skipped). It differs from [D] in two aspects. First, working modulo torsion, we describe mixed sheaves in terms of a canonical "arithmetic" fiber functor to avoid categorical generalities. Second, we use a simple rigidity property of polylogarithm to avoid computations.

Polylogarithm is a special mixed sheaf Π on $\mathbb{P}^1\backslash\{0,1,\infty\}$; the rigidity property claims that Π is completely determined by its most simple quotient–ordinary logarithm (the classical polylogarithm function is just a display of the Hodge version of Π, hence the name). The polylogarithm splits into the sum of k–logarithms $\mathrm{Li}_k(\alpha)$ at these $\alpha \in \mathbb{P}^1\backslash\{0,1,\infty\}$ which are roots of unity.

This picture has absolute motivic counterpart described in n = 5; the corresponding $\mathrm{Li}_k(\alpha)$ are just the cyclotomic elements in rational K–groups. Morally, the rigidity property tells that higher cyclotomic elements are completely determined by the usual, first ones.

Appendix A contains a sketch of iterated integrals construction; as an application we show that the category of lisse mixed sheaves on an algebraic variety X is very much determined by the set of irreducible mixed sheaves and topological fundamental group of X (for unipotent sheaves this fact is equivalent to [HZ]). Appendix B collects some basic information about mixed Tate sheaves in Hodge version.

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1. **Mixed Tate Sheaves.** This section collects some notations and easy general remarks on mixed sheaves.

1.1 Let F be our base field, S := Spec F; we will assume that either $F = \mathbb{C}$ or F is a finite extension of $\mathbb{Q}$. Let X be a smooth scheme of finite type over F; denote by $p : X \to S$ the structure map.

One has at hands the following avatars of a category of mixed lisse sheaves on X.

$\mathcal{M}_{\mathcal{H}_A}(X)$: here $F = \mathbb{C}$, A is either $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$, and $\mathcal{M}_{\mathcal{H}}(X)$:= lisse Hodge sheaves on X = admissible variations of mixed A–Hodge structures (see e.g. [K], or Appendix B)

$\mathcal{M}_{\mathbb{Q}_\ell}(X)$: here F is a number field, and $\mathcal{M}_{\overline{\mathbb{Q}}_\ell}(X)$:= lisse mixed $\overline{\mathbb{Q}}_\ell$–sheaves

$\mathcal{M}_R(X)$: here F is a number field, $\mathcal{M}_R(X)$:= lisse systems of realizations, see [D](1.21).

Below $\mathcal{M}(X)$ will denote either of these categories. So $\mathcal{M}(X)$ is artinian tensor category (A–category in case $\mathcal{M}_{\mathcal{H}_A}$, $\mathbb{Q}_\ell$–one in case $\mathcal{M}_{\overline{\mathbb{Q}}_\ell}$ and $\mathbb{Q}$–one in case $\mathcal{M}_R$). Objects of $\mathcal{M}(X)$ (mixed sheaves) carry a canonical increasing weight filtration W. strictly compatible with any morphism. For a mixed sheaf $\mathcal{F}$, we put $\mathcal{F}_{\leq i} := W_i\mathcal{F}$, $\mathcal{F}_{\geq i} := \mathcal{F}/W_{i-1}\mathcal{F}$, $\mathcal{F}_{[a,b]} := W_a\mathcal{F}/W_{b-1}\mathcal{F}$, $\mathcal{F}_a := \mathcal{F}_{[a,a]} = G^W_a\mathcal{F}$.

A morphism $f : X \to Y$ defines exact tensor "inverse image" functor $f^* : \mathcal{M}(Y) \to \mathcal{M}(X)$. In particular we have $p^* : \mathcal{M}(S) \to \mathcal{M}(X)$; for $G \in \mathcal{M}(S)$ put $G_X := p^*G$. We also have "geometric" cohomology functor $\mathcal{H}^* := R^0p_* : \mathcal{M}(X) \to \mathcal{M}(S)$; the functors p^* and $\mathcal{H}^0 = p_*$ are adjoint.

The simplest mixed sheaves are Tate ones $\mathbb{Q}(i)_\ell$ (we write $\mathbb{Q}(i)$ instead A(i), or $\mathbb{Q}_\ell(i)$, for simplicity of notations). For $\mathcal{F} \in \mathcal{M}(X)$ put $\mathcal{F}(i) := \mathcal{F} \otimes \mathbb{Q}(i)$, $H^0_{\mathcal{M}}(\mathcal{F}) = \mathrm{Hom}(\mathbb{Q}(o)_X, \mathcal{F})$.

A mixed sheaf $\mathcal{F}$ is mixed Tate one if $\mathcal{F}_{2i+1} = 0$ and $\mathcal{F}_{2i}$ is isomorphic to a direct sum of $\mathbb{Q}(-i)$'s for any $i \in \mathbb{Z}$. The category $\mathcal{MT}(X)$ is a full tensor subcategory of $\mathcal{M}(X)$; the functor f^* transforms (mixed) Tate sheaves to (mixed) Tate ones. Assume from now on that X/F is geometrically irreducible; then $\mathcal{H}^0 p^* = \mathrm{id}_{\mathcal{M}(S)}$, p^* is fully faithful and $\mathcal{H}^0(\mathcal{MT}(X)) \subset \mathcal{MT}(S)$. For $\mathcal{F} \in \mathcal{MT}(X)$ put $H^i_{\mathcal{MT}}(\mathcal{F}) := \mathrm{Ext}^i_{\mathcal{MT}}(\mathbb{Q}(o), \mathcal{F})$. We have canonical exact sequence

$$0 \to H^1_{\mathcal{MT}}(\mathcal{H}^0\mathcal{F}) \to H^1_{\mathcal{MT}}(\mathcal{F}) \to H^0_{\mathcal{M}}(\mathcal{H}^1\mathcal{F}) ;$$

denote the image of the last arrow by $H^1_{\mathcal{MT}}(\mathcal{F})^g$ ("geometric part of $H^1_{\mathcal{MT}}$").

1.2 The tensor category $\mathcal{MT}(X)$ has canonical ("arithmetic") fiber functor

$$\phi. = \phi_{X\cdot} : \mathcal{MT}(X) \to (\text{graded vector spaces}),\ \phi_i(\mathcal{F}) := \mathrm{Hom}(\mathbb{Q}(-i), \mathcal{F}_{2i}) = H^0_{\mathcal{MT}}(\mathcal{F}(i)_o)$$

A usual Tannakian story says that $\phi.$ identifies $\mathcal{MT}$ with category L(X).–mod of graded finite–dimensional modules over a graded pronilpotent Lie algebra L(X). ("Fundamental mixed Tate Lie algebra of X".) Explicitly, $L(X)_i$ coincides with the vector space of degree i natural morphisms $\alpha.: \phi. \to \phi_{.+i}$ that satisfy Leibnitz property $\alpha_{\mathcal{F}_1 \otimes \mathcal{F}_2} = \alpha_{\mathcal{F}_1} \otimes \mathrm{id}_{\phi(\mathcal{F}_2)} + \mathrm{id}_{\phi(\mathcal{F}_1)} \otimes \alpha_{\mathcal{F}_2}$. One has $L(X)_i = 0$ for $i \geq 0$ and $L(X)_{-1}$ is dual to the vector space $H^1_{\mathcal{MT}}(\mathbb{Q}(1)_X)$.

For a morphism $f : X \to Y$ one has $\phi_{X.} f^* = \phi_{Y.}$, so we get Lie algebras map $f : L(X) \to L(Y)$ such that $\phi.$ identify f^* with f. change of Lie algebras action functor. The map $p.:L(X). \to L(S).$ is surjective since p^* is fully faithful; put $L(X)^g := \mathrm{Ker}\ p.$ ("geometrically part of L(X).") Clearly $L(X)^g_{-1}$ is dual to the vector space $H^1_{\mathcal{MT}}(\mathbb{Q}(1)_X)^g$. Note that any point $i : S \to X$ defines the splitting $i. : L(S). \to L(X).$ of p.

1.2.1 **Lemma.** Lie algebra $L(X)^g$ is generated by degree -1 component.

Proof. It suffices to show that for any finite dimensional graded L(X).–module F. the subspace of $L(X)^g$–invariant vectors coincides with the one of $L(X)^g_{-1}$–invariants. One has

$F. = \phi.(\mathcal{F})$ for a mixed Tate sheaf $\mathcal{F}_1$ and $F_i^{L(x)^g} = \phi_i \mathcal{H}^0 \mathcal{F}_{\leq 2i}$, $F_i^{L(X)^g_{-1}} = \phi_i \mathcal{H}^0 \mathcal{F}_{[2i,2i-2]}$. So we have to show that projection $\mathcal{F}_{\leq 2i} \to \mathcal{F}_{[2i,2i-2]}$ induces isomorphism on $\phi_i \mathcal{H}^0$. This follows from the exact sequence of cohomology functor $\mathcal{H}^0$ that comes from the short exact sequence $0 \to \mathcal{F}_{\leq 2i-4} \to \mathcal{F}_{\leq 2i} \to \mathcal{F}_{[2i,2i-2]} \to 0$; note that $\mathcal{H}^1 \mathbb{Q}(a)_X$ has weights $-2a+1$, $-2a+2$ only, hence $\mathcal{H}^1 \mathcal{F}_{\leq 2i-4}$ has weights $\leq 2i-2$. □

For a mixed sheaf $\mathcal{F}$ we will call its **geometric data** the graded vector space $\phi.(\mathcal{F})$ considered as $L(X)^g$–module. According to above lemma the $L(X)^g$–action is completely determined by the map $\gamma(\mathcal{F}). : \phi.(\mathcal{F}) \to \phi_{.-1}(\mathcal{F}) \otimes H^1_{\mathcal{MT}}(\mathbb{Q}(1)_X)^g$.

1.2.2 Example. Let $\mathcal{V}$ be a line 1–dimensional F–vector space, $\dot{\mathcal{V}} := \mathcal{V} \setminus \{0\}$. then $H^1_{\mathcal{MT}}(\mathbb{Q}(1)_{\mathcal{V}})^g = 0$, and we have a canonical isomorphism $\mathrm{Res}_0 : H^1_{\mathcal{MT}}(\mathbb{Q}(1)_{\dot{\mathcal{V}}})^g \xrightarrow{\sim} \mathbb{Q}$. Denote the dual map $\mathbb{Q} \xrightarrow{\sim} L(\dot{\mathcal{V}})^g_{-1}$ by $a \longmapsto aN_0$. Since $L(\dot{\mathcal{V}})^g_{-1}$ is one dimensional, 1.2.1 implies that $L(\dot{\mathcal{V}})^g = L(\dot{\mathcal{V}})_{-1} \subset$ center of $L(\dot{\mathcal{V}})$. .

A point $a \in \dot{\mathcal{V}}(F)$ defines the splitting $a. : L(S). \to L(\dot{\mathcal{V}}).$, hence the isomorphism $\tilde{a}. : L(S). \times \mathbb{Q}_{-1} \xrightarrow{\sim} L(\dot{\mathcal{V}})$. We will always identify $L(Gm)$ with $L(S) \times \mathbb{Q}_{-1}$ using $\tilde{1}$; in particular, we will identify mixed Tate sheaves that split at 1 (i.e. with fiber at $1 \in Gm$ isomorphic to direct sum of Tate modules) with graded N_0–modules:= graded vector spaces with degree -1 linear operator N_0. If X is any variety, $\varphi \in \mathcal{O}^*(X)$, then we have a linear map $L(X)_{-1} \xrightarrow{\varphi} L(Gm)_{-1} \xrightarrow{p_2} \mathbb{Q}$, i.e. an element $c\ell(\varphi) \in H^1_{\mathcal{MT}}\mathbb{Q}(1)_X)$. This defines canonical morphisms

$$\begin{array}{ccc} \mathcal{O}^*(X) \otimes \mathbb{Q} & \xrightarrow{c\ell} & H^1_{\mathcal{MT}}(\mathbb{Q}(1)_X) \\ \downarrow & & \downarrow \\ \mathcal{O}^*(X)^g \otimes \mathbb{Q} & \xrightarrow{c\ell^g} & H^1_{\mathcal{MT}}(\mathbb{Q}(1)_X)^g \end{array}$$

where $\mathcal{O}^*(X)^g := \mathcal{O}^*(X)/F^*$. For $\varphi \in \mathcal{O}^*(X)$ we will often write $[\varphi] := c\ell(\varphi)$, $[\varphi]^g := c\ell^g(\varphi)$.

1.3 Let $\bar{X}$ be a smooth curve, $x \in \bar{X}(F)$; put $t_x :=$ tangent space to $\bar{X}$ at x, $j: X := \bar{X} \setminus \{x\} \hookrightarrow \bar{X}$. One has exact tensor functor ("specialization at x") $sp_x : \mathcal{M}(X) \to \mathcal{M}(t_x)$, which transforms mixed Tate sheaves to mixed Tate ones and commutes with ϕ. This defines a canonical Lie algebras morphism $e_x : L(\dot{t}_x) \to L(X)$. Put $N_x := e_x(N_0)$. Note that a mixed Tate sheaf $\mathcal{F}_X$ on X comes from a (unique) sheaf $\mathcal{F}_{\overline{X}}$ on $\overline{X}$ iff $sp_x(\mathcal{F}_X)$ comes from a sheaf on t_x, or, equivalently, if $N_x = 0$ on $\phi.(\mathcal{F}_X)$. This means that $j.: L(X). \to L(\overline{X}).$ identifies $L(\overline{X}).$ with a quotient of $L(X).$ modulo the ideal generated by N_x.

1.3.1 Examples. (i) Assume that $\overline{X} = V$ is one–dimensional vector space, $X = \dot{V}$. The obvious identification $t_0 = \overline{X}$ identifies $sp_0 : \mathcal{MT}(X) \to \mathcal{MT}(\dot{t}_0)$ with identity functor, (ii) Assume that $X = \mathbb{P}^1 \backslash \{x_0,...,x_n\}$, $x_i \in \mathbb{P}^1(F)$. Then N_{x_i} generate $L(X)^g_{-1}$ with the only relation $\Sigma N_{x_i} = 0$. The iterated integrals stuff (see e.g. appendix A) or Deligne's arguments [D] show that $L(X)^g$ is free Lie algebra generated by $L(X)^g_{-1}$.

Remark. Below we will use also mixed Tate sheaves with finiteness condition dropped; these are just arbitrary graded $L(X).$–modules (possibly infinite dimensional).

2. Polylogarithm. Let X be $\mathbb{P}^1 \backslash \{0,1,\infty)$, $T := Gm = \mathbb{P}^1 \backslash \{0,\infty\}$ and $t \in \mathcal{O}^*(T)$ be standard parameter.

2.1 Lemma–definition. There exists a unique mixed Tate sheaf Π on X with geometric data $P_i := \phi(\mathcal{T})_i = \begin{cases} \mathbb{Q} & i \le 0 \\ 0 & i > 0 \end{cases}$, $\gamma(\Pi)_i = \begin{cases} [1\text{-}t]^g & i = 0 \\ [t]^g & i < 0 \end{cases}$. We will call Π (classical) polylogarithm sheaf.

Proof. We have to show that $L(X)^g$–action on the module $P. = \mathbb{Q}_0 \oplus \mathbb{Q}_{-1} \oplus \mathbb{Q}_{-2} \oplus ...$, given by formula $N_0(e_i) = \begin{cases} e_{i-1} & i < 0 \\ 0 & i = 0 \end{cases}$; $N_1(e_0) = e_{-1}$, $N_1(e_i) = 0$ for $i < 0$ (here $e_i = 1 \in \mathbb{Q}_i$), extends to $L(X).$–action in a unique way.

2.1.1 Unicity. Assume we have two such actions $\alpha^1, \alpha^2 : L(X). \to \text{End } P.$ Consider the difference $S := \alpha^1 - \alpha^2 : L(X). \longrightarrow \text{End } P.$ Note that S maps $L(X).$ to $\text{End}_{L(X)^g} P.$ (since S factors through $L(X). \xrightarrow{P.} L(S). \to \text{End } P.$ one has $S(L(X).) = Se_x(L(\dot{t}_x))$ for each $x = 0, 1, \infty$; but $\alpha^i e_x(L(\dot{t}_x))$, hence $Se_x(L(t_x))$, commute with $\alpha^1 N_x = \alpha^2 N_x$). It is easy to see that $\mathrm{End}_{L(X)^g_\cdot} P_\cdot = \mathbb{Q}\cdot \mathrm{id}_{P_\cdot}$. Since $L(X)_\cdot$ supported in negative degrees, one has $S = 0$, i.e. $\alpha_1 = \alpha_2$.

2.1.2 Existence (cf. [D]) Consider $L(X)^g_\cdot$ as $L(X)_\cdot$–module via adjoint action. We will construct $P_\cdot$ as its subquotient. Namely note that $N_0, N_1 \in L(X)^g_{-1}$ generate $L(X)^g_\cdot$ as free Lie algebra (see 1.3.1); one has $[L(X)^g_\cdot, L(X)^g_\cdot] = L(X)^g_{\le -2}$. Put $B_\cdot := \mathbb{Q}\cdot N_0 \oplus L(X)^g_{\le -2}$, $C_\cdot := [L(X)^g_{\le -2}, L(X)^g_{\le -2}] + [N_1, L(X)^g_{\le -2}]$. Clearly $B_\cdot \supset C_\cdot$ are $L(X)_\cdot$–submodules of $L(X)^g_\cdot$. Put $e_0 := N_0$, $e_i := -a\alpha_{N_0}^{-i}(N_i)$ for $i \le -1$. Then $B_{i-1}/C_{i-1} = \mathbb{Q}\, e_i$, and $P_\cdot = B_{\cdot -1}/C_{\cdot -1}$ is desired $L(X)_\cdot$–module. □

2.1.1 Remark. Actually the proof of unicity shows that if $\mathcal{F}$ is any mixed Tate sheaf on X and $\alpha : \phi_\cdot(\mathcal{F}_{\ge 2i}) \longrightarrow \phi_\cdot(\Pi_{\ge 2i})$ is an isomorphism of geometric data, then α defines an isomorphism $\mathcal{F}_{\ge 2i+2} \longrightarrow \Pi_{\ge 2i+2}$ of mixed Tate sheaves. □

Let R be mixed Tate sheaf on T with geometric data $R_i = \phi_i(R) = \mathbb{Q}$ if $i \le 0$, $R_i = 0$ for $i > 0$, $\gamma(R)_i = [t]^g$, and such that R splits at $t = 1$ (in notations of 1.2.2 R corresponds to the graded N_0–module $\mathbb{Q}_0 \xrightarrow{N_0} \mathbb{Q}_{-1} \xrightarrow{N_0} \mathbb{Q}_{-2} \longrightarrow \dots$, $N_0 = 1$). Clearly one may identify $R_{\ge -2i}$ with symmetric power $\mathrm{sym}^i R_{\ge -2}$.

2.2 Lemma. (i) The obvious isomorphism of geometric data $\phi_\cdot(\Pi_{\le -2}) \xrightarrow{\sim} \phi_\cdot(R_X(1))$ comes from an isomorphism of mixed Tate sheaves $\Pi_{\le -2} \xrightarrow{\sim} R_X(1)$.

(ii) $Sp_0(\Pi) = \mathbb{Q}(0)_{\dot{t}_0} \oplus Sp_0(R(1))$.

Proof. (i) We have to show that $\Pi_{\le -2}$ extends to T and has split fiber at $1 \in T$. The first fact is clear, since N_1 acts trivially on $P_{\le -1}$. It remains to show that $L(S)_\cdot$ acts trivially on its fiber at 1, or that $L(\dot{t}_1)$ acts trivially $P_{\le -1}$. Note that $L(\dot{t}_1)$ kills e_{-1}: for $\ell \in L(\dot{t}_1)$ one has $\ell e_{-1} = \ell N_1 e_0 = N_1 \ell e_0 = 0$, since $\ell_1 e_0 \in P_{\le -1}$ and $N_1 P_{\le -1} = 0$. Now $L(S)_\cdot$–action on $P_{\le -1}$ commutes with N_0–action (see 1.2.2), hence $L(S)_\cdot e_i = L(S)_\cdot N_0^{-i-1} e_{-1} = N_0^{-i-1} L(S_\cdot) e_{-1} = 0$.

(ii) We have to show that $L(\dot{t}_0)$–action kills e_0. Since $L(\dot{t}_0)e_0 \subset P_{\le -1}$ and N_0 acts injectively on $P_{\le -1}$ it suffices to show that $N_0 L(\dot{t}_0) e_0 = 0$. But N_0 lies in center of $L(\dot{t}_0)$, hence $N_0 L(\dot{t}_0) e_0 = L(\dot{t}_0) N_0 e_0 = 0$. Another proof. Look at construction 2.1.2: one has $e_0 = N_0$, and adjoint action of $L(\dot{t}_0)$ kills N_0. □

2.2.1 Corollary. The class of $\Pi_{[-2i,-2i-2]}$ in $\mathrm{Ext}^1_{\mathcal{MT}}(\mathbb{Q}(i)_x, \mathbb{Q}(i+1)_x)$ is $[1-t]$ for $i = 0$ and $[t]$ for $i > 0$, so conditions 2.1 on $\gamma(\Pi)$ hold also on "arithmetic" level.

2.3 Definition. Polylogarithm is the class in $H^1_{\mathcal{MT}}(\mathbb{Q}(0), R_X(1))$ of our sheaf Π.

Note that this class determines Π up to a canonical isomorphism, so we will denote it by the same letter Π.

3. Formulas. Let us describe Π in explicit terms.

3.1 The $\mathbb{Q}$–Hodge avatar of Π is as follows. Its holomorphic data is graded vector bundle $\underset{i\le 0}{\oplus} \mathcal{O}_X e_i$ with connection $\nabla : \nabla(e_i) = \frac{dt}{t} e_{i-1}$ for $i \le -1$, $\nabla(e_0) = \frac{dt}{t-1} e_{-1}$ (as usually, see appendix B, the Hodge filtration is $F^{\cdot} = \underset{j\ge \cdot}{\oplus} \mathcal{O}_X e_j$ and weight one W_{2i} is $\underset{j\le i}{\oplus} \mathcal{O}_X e_j$). The $\mathbb{Q}$–structure on Π^∇ may be found from (2.2): it is formed by $\mathbb{Q}$–linear combinations of multivalued sections $e_0 + \sum_{k\ge 1} Li_k(t)\cdot e_{-k}$, $(2\pi\sqrt{-1})^i \sum_{k\ge 0} \frac{(-1)^k \log^k t}{k!} e_{-i-k}$ $(i \ge 1)$; here $Li_k(t) = \sum_{n>0} \frac{t^n}{n^k}$ is classical k–logarithm.

3.2 Let us spell $\mathbb{R}$–Hodge version of Π in language of appendix B2. Our Π is graded vector space $P. = \underset{i\le 0}{\oplus} \mathbb{C}\, e_i$ equipped with real structure $P_{\mathbb{R}}. = \underset{i\le 0}{\oplus} (2\pi\sqrt{-1})^{-i}\, \mathbb{R}\, e_i$. The C^∞–function $T : X \to \mathrm{Aut}\, P.$ is given by formula $T(e_i) = \sum_{k\ge 0} \frac{(-1)^k}{k!} \log|t|^2 e_{i-k}$ for $i \le -1$ and $T(e_0) = e_0 + \sum_{k\ge 1} \Big(Li_k(t) - (-1)^k \sum_{\substack{a,\, b\ge 0\\ a+b=k}} \overline{Li}_a(t) \frac{\log^b|t|^2}{b!}\Big)$. Note that $T = \exp N$, where $N(e_i) = -\log|t|^2 e_{i-1}$ for $i \le -1$ and $N(e_0) = \sum_{j\ge 1} D_j(t) e_{-j}$, where $\sum_{j\ge 1} D_j(t) q^j =$ $\Big[\sum_{k\ge 1} Li_k(t) q^k - \Big(\sum_{k\ge 1} \overline{Li}_k(t)(-q)^k\Big)\Big(\sum_{i\ge 1} \frac{|t|^{2i} q^i}{i!}\Big)\Big]\Big(\sum_{k\ge 1} \frac{B_k}{k!} \log^k|t|^2 q^k\Big)$. This D_j is just single valued version of

B_k are Bernoulli numbers

polylogarithm found by Bloch and Wiegner in case $j = 2$, and by Ramakrishnan and Zagier in general case [R], [Z].

3.3 To describe $\mathbb{Q}_\ell$–version of polylogarithm (cf. [D]) we need to fix some notations.

3.3.1 Let K be a finite set, A be an abelian group. Put $A[K] = A^K$: we will consider $A[K]$ as the group of A–valued measures on K. Notation: $a = \sum_{k \in K} a_k \delta_k \in A[K]$.

Let $\mathcal{E}_{A,K}$ be an A–torsor over K: so we have surjective map of sets $\pi : \mathcal{E}_{AK} \to K$ with simple transitive A–action along the fibers $\mathcal{E}_{AK}(k) := \pi^{-1}(k)$ of π. Denote by $\Gamma(\mathcal{E}_{AK})$ the set of sections of π; this is $A[K]$–torsor with respect to $A[K]$–action defined by formula $(a \cdot \gamma)(k) = a_k \gamma(k)$. The functor Γ : (A–torsors over K) $\to$ (A[K]–torsors) is equivalence of categories.

Let $f : K_2 \to K_1$ be a mapping of sets, $\mathcal{E}_{AK_i}$ be torsors over K_i. An f–morphism $\tilde{f} : \mathcal{E}_{AK_2} \to \mathcal{E}_{AK_1}$ is, by definition, a collection of maps $\tilde{f}(k_1) : \prod_{k_2 \in f^{-1}(k_1)} \mathcal{E}_{AK_2}(k_2) \to \mathcal{E}_{AK_1}(k)$, $k_1 \in K_1$, such that one has $\tilde{f}(k_1)(\Pi a_{k_2} e_{k_2}) = \Pi a_{k_2} \tilde{f}(k_1)(\Pi e_{k_2})$ (if $f^{-1}(k_1)$ is empty, then $\tilde{f}(k_1)$ fixes a point in $\mathcal{E}_{AK_1}(k_1)$). Equivalently, we have direct image functor $f_!$: (A–torsors over K_2) $\to$ (A–torsors over K_1) defined by formula $(f_! \mathcal{E}_{AK_2})(k_1) = \prod_{k_2 \in f^{-1}(k_2)} \mathcal{E}_{AK_2}(k_2)$, where "$\Pi$" is product of A–torsors, and f–morphism $\tilde{f}$ is just a morphism $f_! \mathcal{E}_{AK_2} \to \mathcal{E}_{AK_1}$. Note that f defines "integration along the fibers" map $f_\cdot$: $A[K_2] \to A[K_1]$, $f_\cdot(\Sigma\, a_{k_2} \delta_{k_2}) = \Sigma\, a_{k_2} \delta_{f(k_2)}$, and $\tilde{f} : \mathcal{E}_{AK_2} \to \mathcal{E}_{AK_1}$ defines $f_\cdot$–morphism of $A[K_i]$–torsors $\Gamma(\tilde{f}) : \Gamma(\mathcal{E}_{AK_2}) \to \Gamma(\mathcal{E}_{AK_1})$.

Assume we have a projective system of sets... $\to K_3 \xrightarrow{\mu} K_2 \xrightarrow{\mu} K_1$ and corresponding projective system of A–torsors ... $\to \mathcal{E}_{AK_3} \xrightarrow{\tilde{\mu}} \mathcal{E}_{AK_2} \xrightarrow{\tilde{\mu}} \mathcal{E}_{AK_1}$. Then we have projective limits: $K = \varprojlim K$, $A[[K]] = \varprojlim A[K_i]$ = A–valued measures on K, and $A[[K]]$–torsor $\Gamma(\mathcal{E}_{AK}) := \varprojlim \Gamma(\mathcal{E}_{AK_i})$.

All these constructions are obviously compatible with change of coefficients by morphisms $A \to A'$.

3.3.2 Let us apply this general stuff to our situation. Fix a prime ℓ. For a point $\alpha \in T = Gm$ and $n \geq 1$ consider a set $K_{n,\alpha} := \{\beta : \beta^{\ell^n} = \alpha\}$ of ℓ^n–roots of α. These $K_{n,\alpha}$ form a compatible system of $\mathbb{Z}/\ell^n(1)$–torsors with respect to maps $\mu : K_{n,\alpha} \to K_{n-1,\alpha}$, $\mu(\beta) = \beta^\ell$, hence we have a $\mathbb{Z}_\ell(1)$–torsor $K_\alpha := \varprojlim K_{n,\alpha}$. Now for $m \geq 1$, (assuming that $\alpha \neq 1$) consider $\mathbb{Z}/\ell^n(1)$–torsors $\mathcal{E}^{(m)}_{K_{n,\alpha}}$ over $K_{n,\alpha}$ with fibers $\mathcal{E}^{(m)}_{K_{n,\alpha}}(\beta) := \{\gamma : \gamma^{\ell^m} = 1 - \beta\} = K_{m,1-\beta}$. We have a system of μ–morphisms $\tilde{\mu} : \mathcal{E}^{(m)}_{K_{n,\alpha}} \to \mathcal{E}^{(m)}_{K_{n-1,\alpha}}$, $\tilde{\mu}(\beta)(\prod_{\delta:\, \delta^\ell = \beta} \gamma_\delta) := \prod \gamma_\delta \in \mathcal{E}^{(m)}_{K_{n-1,\alpha}}(\beta)$ (here $\beta \in K_{n-1,\alpha}$, $\delta \in \mu^{-1}(\beta) \subset K_{n,\alpha}$, $\gamma_\delta \in \mathcal{E}^{(m)}_{K_{n,\alpha}}(\delta)$. These define $\mathbb{Z}/\ell^m(1)[[K_\alpha]] = \mathbb{Z}/\ell^m[[K_\alpha]](1)$–torsor $\Gamma(\mathcal{E}^{(m)}_{K\alpha})$. Note that $\mathcal{E}^{(m)}_{K_{N,\alpha}}$ form compatible system of $\mathbb{Z}/\ell^m(1)$–torsors (with respect to maps $\mathcal{E}^{(m)} \to \mathcal{E}^{(m-1)}$, $\gamma \longmapsto \gamma^\ell$), hence we get a projective limit $\Gamma(\mathcal{E}_{K_\alpha}) = \varprojlim \Gamma(\mathcal{E}^{(m)}_{K_\alpha})$ which is $\mathbb{Z}_\ell[[K_\alpha]](1)$–torsor (here $\mathbb{Z}_\ell[[K_\alpha]] := \varprojlim \mathbb{Z}/\ell^m[[K_\alpha]]$).

Recall that K_α is $\mathbb{Z}_\ell(1)$–torsor, hence $R_\alpha := \mathbb{Z}_\ell[[K_\alpha]]$ is a free rank 1 module over (completed) group ring (Iwasawa algebra) $\mathcal{I} := \mathbb{Z}_\ell[[\mathbb{Z}_\ell(1)]]$ = algebra of $\mathbb{Z}_\ell$–valued measures on $\mathbb{Z}_\ell(1)$. Let $I \subset \mathcal{I}$ be augmentation ideal; $\mathcal{I}$ is complete with respect to I–adic filtration: $\mathcal{I} = \varprojlim \mathcal{I}/I^n$. One has canonical isomorphisms $\mathbb{Z}_\ell(n) \xrightarrow{\sim} I^n/I^{n+1}$, $u^{\otimes n} \longmapsto (u-1)^n$, $u \in \mathbb{Z}_\ell(1)$, so the graded ring $gr_I \mathcal{I} = \oplus I^n/I^{n+1}$ is just a polynomial ring $\mathbb{Z}_\ell[u]$, $u \in \mathbb{Z}_\ell(1)$. If we extend coefficients to $\mathbb{Q}_\ell$, then we get a canonical isomorphism $\mathbb{Q}_\ell[[u]] \xrightarrow{\sim} \mathcal{I}_{\mathbb{Q}_\ell} := \mathbb{Q}_\ell \hat{\otimes} \mathcal{I} = \varprojlim \mathbb{Q}_\ell \otimes \mathcal{I}/I^n$, defined by formula $u \longmapsto \log \delta_u = -\Sigma \frac{(1-\delta_u)^n}{n}$; the inverse isomorphism $\mathcal{I}_{\mathbb{Q}_\ell} \to \mathbb{Q}_\ell[[u]]$ is moment map $\mu \longmapsto \sum_{n\geq 0} (\int_{\mathbb{Z}_\ell(1)} u^{-n}\epsilon^n \mu) \frac{u^n}{n!}$ (here $\mu \in \mathcal{I}_{\mathbb{Q}_\ell}$ is a $\mathbb{Q}_\ell$–valued measure on $\mathbb{Z}_\ell(1)$, $\epsilon = id_{\mathbb{Z}_\ell(1)}$ so $u^{-n}\epsilon^n$ is $\mathbb{Z}_\ell$–valued function on $\mathbb{Z}_\ell(1)$). Consider how I–adic filtration on R_α. Since $R_\alpha/I = \mathbb{Z}_\ell$, one has a canonical isomorphism

$I^nR_\alpha/I^{n+1}R_\alpha = \mathbb{Z}_\ell(n)$, and $R_{\alpha_{\mathbb{Q}_\ell}}$ is free rank 1 module over $\mathbb{Q}_\ell[[u]]$. For $\alpha = 1$, we have a canonical identification of R_1 with $\mathcal{I}$ (since $K_1 = \mathbb{Z}_\ell(1)$).

When $\alpha \in T$ varies, the above R_α forms $\mathbb{Z}_\ell$–sheaf R over T. The above arguments show that corresponding $\mathbb{Q}_\ell$–sheaf $R_{\mathbb{Q}_\ell}$ is canonically isomorphic to same noted sheaf from 2.2(i). Same way, $\Gamma(\mathcal{E}_{K_\alpha})$ are fibers of R(1)–torsor $\Gamma(\mathcal{E}_K)$ over X, which gives rise to an $R_{\mathbb{Q}_\ell}(1)$–torsor $\Gamma(\mathcal{E}_K)_{\mathbb{Q}_\ell}$. But $R_{\mathbb{Q}_\ell}(1)$–torsor is the same thing as extension $0 \to R_{\mathbb{Q}_\ell}(1) \to \mathcal{L}i \to \mathbb{Q}_\ell(0) \to 0$. An easy computation shows that this $\mathcal{L}i$ satisfies 2.1, hence coincides with $\mathbb{Q}_\ell$–version of polylogarithm sheaf.

4. Cyclotomic elements. Let $\alpha \in F$ be a degree in root of unity: $\alpha^m = 1$. Consider the point $t = \alpha$ of T. The sheaf R splits over α (since $R_{\geq -2}$ does, and $R_{\geq -2i} = \mathrm{Sym}^i R_{\geq -2}$), so we have a canonical decomposition of fiber $R_\alpha(1) = i_\alpha^* R(1) = \mathbb{Q}(1)_S \oplus \mathbb{Q}(2)_S \oplus \ldots$. Denote by $pr_\alpha : R_\alpha(1) \to \mathbb{Q}(k)_S$ the n^{th} projection. Let $Li_k(\alpha) := pr_{k,\alpha}(\Pi_\alpha) \in H^1_{\mathcal{M}}(\mathbb{Q}(k)_S$ be the k^{th} component of polylogarithm at α: this is the desired cyclotomic element.

4.1 As follows from 3.1 the Hodge version of $Li_k(\alpha) \longleftrightarrow \in \mathbb{C}/(2\pi\sqrt{-1})^n \cdot \mathbb{Q} = H^1_{\mathcal{M}_\mathcal{H}}(\mathbb{Q}(n))$ is just the value of α at classical k–logarithm function. In $\mathbb{Q}_\ell$–version the projections $pr_{k,\alpha} : R_{\mathbb{Q}_\ell\alpha}(1) \to \mathbb{Q}_\ell(n)$ are given by formula $pr_{k\alpha}(\mu) = \frac{1}{k!}\int \epsilon_m^k \mu$, where $\mu \in R_{\mathbb{Q}_\ell\alpha}(1)$ is $\mathbb{Q}_\ell(1)$–valued measure on K_α, and $\epsilon_m : K_\alpha \xrightarrow{\sim} \mathbb{Z}_\ell(1)$ transforms $\beta = \varprojlim \beta_n$ to $\mathcal{E}_m(\beta) = \varprojlim \beta_n^m$. To get cyclotomic units one should push out the class of $R_\alpha(1)$–torsor $\mathcal{E}_{K_\alpha}$ to $\mathbb{Q}_\ell(k)$–torsor by means of $pr_{k\alpha}$: these are just the Galois cohomology classes in $H^1(\mathrm{Gal}\ \bar{F}/F, \mathbb{Q}_\ell(k))$ defined by Soulé [S] and Deligne [D].

4.2 Remark. One may describe a canonical splitting $R_\alpha \simeq \oplus \mathbb{Q}(n)_S$ in another way. For an integer a consider morphism $\mu_a : T \to T$, $\mu_a(t) = t^a$. The sheaf $\mu_a^* R$ splits at $t = 1$, and with respect to canonical isomorphisms $Gr^W_{2n}\mu_a^* R = \mu_a^* Gr^W_{2n} R = \mathbb{Q}(-n)_X$ the classes of $\mu_a^* R_{[-2n,-2n-2]}$ are $\mu_a^*[t] = [t^a]$ (see 2.2.1). These properties determine $\mu_a^*(R)$ uniquely, hence one has (unique) morphism $\tilde{\mu}_a : R \to \mu_a^* R$ such that $G^W_{-2n}(\tilde{\mu}_a)$ is multiplication by a^n. Now choose $a \in \mathbb{Z}$ such that $a = 1 \bmod m$. Then $\mu_a(\alpha) = \alpha$, hence $\tilde{\mu}_a$ acts on the fiber R_α. If $a \neq 1$ then $\mathbb{Q}(n)_S \subset R_\alpha$ is just the eigenspace of $\tilde{\mu}_a$ with eigenvalue a^n.

5. **Motivic Story.** Though up to now we do not know what mixed motives are, we may rephrase the above constructions in the language of absolute motivic cohomology $H^{\cdot}_{\mu}(?,\mathbb{Q}(*))$ defined by means of algebraic K–theory (see e.g. [B]). More precisely, the group $H^1_{\mathcal{MT}}(X,R_X(1))$, where polylogarithm Π lives, is actually an absolute cohomology group with "constant" coefficients of certain simplicial scheme Y. We may consider instead the corresponding absolute motivic cohomology of Y., and find there a canonical element Π_{mot} (motivic polylogarithm) whose various realizations (or regulators) are $\mathbb{Q}_\ell$– and Hodge versions of polylogarithm from §2. The values of Π_{mot} at roots of unity are cyclotomic elements in absolute motivic cohomology.

To define R in geometric ("motivic") way one may use iterated integrals. The construction goes as follows (for a general construction see Appendix A).

5.1. Let us define for $n \geq 1$ the augmented simplicial T–scheme $Y^{(n)}_{\cdot}$. Let T^{n+1} be n+1–dimensional torus with coordinate functions $x_0,\dots,x_n$. Denote by $y_0,\dots,y_n$ the new coordinate functions $y_i = x_i/x_{i+1}$ for $i \neq n$, $y_n = x_n$, so $x_i = y_i y_{i+1} \cdot \ldots \cdot y_n$. For a subset $A \subset \{0,\dots,n\}$, $A \neq \{0,\dots,n\}$, let $Y^{(n)}_A \subset T^{n+1}$ be the subscheme defined by equations $y_i = 1$ for $i \in A$. If $A = \{i_0,\dots,i_a\}$, $i_0 < \dots < i_a$, we put $A_{(j)} = A \backslash \{i_j\}$ and denote by $\partial_j : Y^{(n)}_A \to Y^{(n)}_{A(j)}$ an obvious embedding.

For an integer $i \geq -1$ define a T–scheme $Y^{(n)}_{i(nd)}$ to be disjoint union of $Y^{(n)}_A$, $A \subset \{0,\dots,n\}$, $|A| = i + 1$ if $-1 < i < n$; if $i \geq n$ then $Y^{(n)}_{i(nd)} = \emptyset$; the structure map $Y^{(n)}_{\cdot(nd)} \to T$ is $t = x_0$. We may consider ∂_j as boundary maps, and will define our augmented simplicial T–scheme $Y^{(n)}_{\cdot}$ as the one obtained by a standard universal construction from its variety of non–degenerate simplices $Y^{(n)}_{.(nd)}$. For any T–scheme $t : U \to T$ (so $t \in \mathcal{O}^*(U)$) put $Y^{(n)}_{Ut.} := U \underset{T}{\times} Y^{(n)}_{.}$, so $Y^{(n)}_{Ut-1} = U \times T^n$, etc.

5.2 We are going to compute $H^{\cdot}_{\mu}(Y^{(n)}_{Ut.}, \mathbb{Q}(*))$.

Remark. Below $H^{\cdot}_{\mu}(?, \mathbb{Q}(*))$ denote the absolute motivic cohomology constructed by means of Quillen's K–groups. While computing the cohomology we will consider any augmented simplicial scheme Y. as space Y_{-1} modulo subspace $Y_{\geq 0}$ i.e. we put degrees in cohomology in a way that one has exact sequence $\dots \to H^{\cdot}(Y.) \to H^{\cdot}(Y_{-1}) \to H^{\cdot}(Y_{\geq 0}) \to \dots$.

The following notation will be convenient.

For any $n \geq 1$ denote by $S(-n).$ the following augmented simplicial S–scheme. Let T^n be n–dimensional torus with coordinates $y_1,\dots,y_n$; for $A \subset \{1,\dots,n\}$ let $S(-n)_A$ be subscheme of T^n defined by equations $y_i = 1$, $i \in A$; let $\partial_j : S(-n)_A \to S(-n)_{A(j)}$ be obvious embeddings. Let $S(-n)_{i(nd)}$ be disjoint union of $S(-n)_A$, $|A| = i+1$; then ∂_j is a system of boundary maps, and we define $S(-n).$ as simplicial scheme obtained from its non–degenerate simplices $S(-n)_{.(nd)}$ by universal construction. For any scheme U put $U(-n) := U \underset{S}{\times} S(-n)$. Assume from now on that U is regular. Then, according to Quillen, one has a canonical isomorphism $H^{\cdot}_{\mu}(U,\mathbb{Q}(*) \longrightarrow H^{\cdot+n}_{\mu}(U(-n), \mathbb{Q}(*+n))$ defined by formula $a \longmapsto a \cup y_1 \cup\dots\cup y_n$, where y_i are coordinate functions considered as elements of $H^1_{\mu}(U(-n), \mathbb{Q}(1))$ (and $\cup$ is cup–product on absolute motivic cohomology). This explains the notation.

5.3. Consider the maps $Y^{(n-1)}_{Ut.} \xrightarrow{i_n} Y^{(n)}_{Ut}$ of agumented simplicial U–schemes defined on 1–simplices by formulas $i_n(x_0,\dots,x_{n-1}) = (y_1,\dots,y_n)$, where $y_i = x_{i-1}$, $j_n(y_1,\dots,y_n) = (x_0,\dots,x_n)$, where $x_0 = t$, $x_i = \prod_{j\geq i} y_j$ for $i \geq 1$, and such that i_n transforms the component $Y^{(n-1)}_B$, $B \subset \{0,\dots,n-1\}$, to $U(-n)_{B'}$, $B' = \{\alpha+1 : \alpha \in B\} \subset \{1,\dots,n\}$, and j_n transforms $U(-n)_A$ to $Y^{(n)}_A$.

As follows directly from definitions j_n identifies $Y_{Ut}^{(n)}$ with Cone(i_n). Hence one gets exact cohomology sequence $\ldots \to H^{\cdot}_{\mathcal{M}}(Y_{Ut}^{(n)}, \mathbb{Q}(x)) \xrightarrow{j_n^{\cdot}} H^{\cdot}_{\mathcal{M}}(U(-n), \mathbb{Q}(*) = H^{\cdot-n}_{\mathcal{M}}(U, \mathbb{Q}(*-n)) \xrightarrow{i_n^{\cdot}} H^{\cdot}_{\mathcal{M}}(Y_{Ut}^{(n-1)}, \mathbb{Q}(*)) \to \ldots$. Walking downwards by n we get the spectral sequence with terms $E_1^{p,q} = H_{\mathcal{M}}^{p+q}(U, \mathbb{Q}(*+p))$, $p = 0,\ldots,n$, that converges to $H_{\mathcal{M}}^{p+q+n}(Y_{Ut}^{(n)}, \mathbb{Q}(*+n))$. The differential $d_1 : E_1^{p,q} \to E_1^{p+1,q}$ is just $\cup$ t. (Proof: d_1 is just the composition $H^{\cdot}_{\mathcal{M}}(U,\mathbb{Q}(*)) \to H^{\cdot+p}_{\mathcal{M}}(U(-p), \mathbb{Q}(*+p)) \xrightarrow{(i_p j_{p-1})^*} H^{\cdot+p}_{\mathcal{M}}(U(-p+1), \mathbb{Q}(*+p)) \leftarrow H^{*+1}_{\mathcal{M}}(U, \mathbb{Q}(*+1))$; since $i_p j_{p-1}(y_1,\ldots,y_{p-1}) = (t, y_{1_\ell}\ldots y_{p-1},\ldots,y_n)$, and $y_1 \cup\ldots\cup y_{p-1} = (y_1\ldots y_{p-1}) \cup (y_2\ldots y_{p-1}) \cup \ldots \cup U_{p-1}$, we are done).

Here are basic properties of this spectral sequence.

5.3.1. Compatibility with localization. U be a curve, $x : S'' \to U$ be a point; put $V = U\backslash x(S)$. We have exact localization sequence $\ldots \to H^{\cdot+p}_{\mathcal{M}}(Y_{U,t,}^{(n)} \mathbb{Q}(*+n)) \to H^{\cdot+n}_{\mathcal{M}}(Y_{V,t,}^{(n)} \mathbb{Q}(*+n)) \to H^{\cdot+n-1}(Y_{S,t(x)}^{(n)}, \mathbb{Q}(*+n-1)) \to \ldots$ together with corresponding exact sequence of $E_r^{p,q}$; for r = 1 this coincides with a usual localization sequence $\ldots \to H^{\cdot}_{\mathcal{M}}(U,\mathbb{Q}(*)) \to H^{\cdot}_{\mathcal{M}}(V,\mathbb{Q}(*)) \xrightarrow{Res_x} H^{\cdot-1}_{\mathcal{M}}(S,\mathbb{Q}(*-1)) \to \ldots$

5.3.2. Change of t. For $a \in \mathbb{Z}$ consider a morphism of augmented simplicial U–schemes $\mu_a^{(n)} : Y_{U,t}^{(n)} \to Y_{U,t^a}^{(n)}$ defined by formula $\mu_a^{(n)}(x_0,\ldots,x_n) = (x_0^a,\ldots,x_n^a)$ (so $\mu_a^{(n)}$ transforms $Y_A^{(n)}$ to $Y_A^{(n)}$, see 5.1). One has the corresponding morphism of spectral sequences; on $E_1^{p,q} = H_{\mathcal{M}}^{p+q}(U, \mathbb{Q}(*+p))$ it is just multiplication by a^{n-p} (this follows directly from construction of spectral sequence).

5.3.3. Degeneration of roots of unity. If t is root of unity, then the spectral sequence degenerates at E_1. To see this just choose $a \neq 1$ such that $t^a = t$; then $\mu_a^{(n)}$ acts on our spectral sequence with eigenvalues a^{n-p} on $E_2^{p,q}$. Hence $d_r = 0$ for $r \geq 1$. Moreover, decomposition of $H^{\cdot+n}_{\mathcal{M}}(Y_{U,t,}^{(n)} \mathbb{Q}(*+n))$ by eigenspaces of $\mu_a^{(n)}$ determines a canonical isomorphism $H_{\mathcal{M}}^{\cdot+n}(Y_{U,t,}^{(n)}, \mathbb{Q}(*+n)) = \bigoplus_{0\le i\le n} H_{\mathcal{M}}^{\cdot}(S, \mathbb{Q}(*+i))$ (since $\mu_a^{(n)}$'s commute, this decomposition does not depend on the choice of a).

Now let us consider the case when our U is $X := P^1\setminus\{0,1,\infty\} \hookrightarrow T$. Assume also that our base field is number field. The following basic lemma is an analog of 2.1.

5.4 Lemma (i) The sequence $0 \to H_{\mathcal{M}}^1(S,\mathbb{Q}(n+1)) \xrightarrow{\alpha_n p^*} H_{\mathcal{M}}^{n+1}(Y_{Xt}^{(n)}, \mathbb{Q}(n+1)) \xrightarrow{\beta_n} H_{\mathcal{M}}^1(X, \mathbb{Q}(1))$ is exact. Here α_n, β_n are edge homomorphisms of above spectral sequence.
(ii) The image of β_n is subspace of $H_{\mathcal{M}}^1(X, \mathbb{Q}(1)) = \mathcal{O}^*(X) \otimes \mathbb{Q}$ generated by t and 1–t.
Proof. According to Borel and Quillen for $i \ge 2$ one has isomorphisms $p^*: H_{\mathcal{M}}^1(S,\mathbb{Q}(i)) \longrightarrow H_{\mathcal{M}}^1(X, \mathbb{Q}(i))$, $a : H_{\mathcal{M}}^1(S, \mathbb{Q}(i-1)) \oplus H_{\mathcal{M}}^1(S, \mathbb{Q}(i-1) \longrightarrow H_{\mathcal{M}}^2(X, \mathbb{Q}(i))$, where $a(\ell_1,\ell_2) := P^*(\ell_1) \cup t + P^*(\ell_2) \cup (1-t)$ (and $t, 1-t \in \mathcal{O}^*(X) \otimes \mathbb{Q} = H_{\mathcal{M}}^1(X, \mathbb{Q}(1))$; the cohomology groups $H_{\mathcal{M}}^j(X, \mathbb{Q}(i))$ for $j \ne 1,2$, $i \ne 0$ vanish. Note that the inverse map $a^{-1} : H_{\mathcal{M}}^2(X, \mathbb{Q}(i)) \longrightarrow H_{\mathcal{M}}^1(S, \mathbb{Q}(i-1)) \oplus H_{\mathcal{M}}^1(S, \mathbb{Q}(i-1))$ is $a^{-1}(m) = (\mathrm{Res}_0(m), \mathrm{Res}_1(m))$.

This implies that the only non–zero terms of the spectral sequence, that computes $H_{\mathcal{M}}^{n+\cdot}(Y^{(n)}, \mathbb{Q}(n+1))$, are $E_1^{p,1-p} = H_{\mathcal{M}}^1(X, \mathbb{Q}(p+1))$, $E_1^{p,2-p} = H_{\mathcal{M}}^2(X, \mathbb{Q}(p+1))$, $p = 0,\dots,n$; the differential $d_1 : E_1^{p,1-p} \to E_1^{p+1,1-p}$ is $\cup t$. The composition $H_{\mathcal{M}}^1(S,\mathbb{Q}(p+1) \xrightarrow{p^*} E_1^{p,1-p} \xrightarrow{d_1} E_1^{p+1,1-p} \xrightarrow{\mathrm{Res}_0} H_{\mathcal{M}}^1(S,\mathbb{Q}(p+1))$ is identity map (for $p = 0,\dots,n-1$), and $\mathrm{Res}_1 d_1 = 0$. Since p^* is isomorphism for $p = 1,\dots,n-1$, the above isomorphism a^{-1} shows that for these p we have short exact sequence

$$0 \to E_1^{p,1-p} \xrightarrow{d_1} E_1^{p+1,1-p} \xrightarrow{\mathrm{Res}_1} H_{\mathcal{M}}^1(S,\mathbb{Q}(p+1)) \to 0$$

For $p = 0$, d_1 is not injective, and we have exact sequence

$$E_1^{0,1} \xrightarrow{d_1^{0,1}} E_1^{1,1} \xrightarrow{\mathrm{Res}_1} H_{\mathcal{M}}^1(S,\mathbb{Q}(p+1)) \to 0$$

with $\mathrm{Ker}\, d_1 \subset E_1^{01} = \mathcal{O}^*(X) \otimes \mathbb{Q}$ equal to subspace ϕ generated by t, $1-t \in \mathcal{O}^*(X)$ (clearly $\phi \subset \mathrm{Ker}\, d_1$ by Steinberg identity; since $\mathcal{O}^*(X) \otimes \mathbb{Q} = \phi \oplus \mathcal{O}^*(S) \otimes \mathbb{Q}$ and $\mathrm{Res}_0\, d_1$ is identity on second term, we get $\mathrm{Ker}\, d_1 = \phi$).

This means that the only non–zero $E_2^{p,q}$'s are $E_2^{n,1-n} = H^1_{\mathcal{M}}(S, \mathbb{Q}(n+1))$, $E_2^{0,1} = \phi$, and $E_2^{p,2-p} \xrightarrow[\sim]{Res_1} H^1_{\mathcal{M}}(S, \mathbb{Q}(p))$, $p = 1,\ldots,n$. This implies 5.5(i). To prove 5.5(ii) one has to show that our spectral sequence degenerates at E_2, i.e. that all higher differentials d_r : $E_r^{0,1} \longrightarrow E_r^{r,2-r}$ vanish for $r \geq 2$. Using induction by r we may assume that $Res_1 : E_2^{r,2-r} \to H^1_{\mathcal{M}}(S,\mathbb{Q}(r))$ is isomorphism. But $Res_1 d_r = d_r^{(1)} Res$, where $d_r^{(1)}$ is the differential of spectral sequence for $H^{\cdot+n}(Y^{(n)}_{S,1}, \mathbb{Q}(*+n))$ (see 5.3.1). Since the last spectral sequence degenerates at E_1 by 5.3.3, one has $d_r^{(1)} = 0$, hence $d_r = 0$, and we are done. □

5.5. Define motivic polylogarithm $\Pi_{mot} \in H^{n+1}_{\mathcal{M}}(Y^{(n)}_X, \quad \mathbb{Q}(n+1))/H^1_{\mathcal{M}}(S, \mathbb{Q}(n+1))$ to be a unique element that maps to $1-t \in H^1(X, \mathbb{Q}(1))$ by β_n (see 5.4).

Remark. One may identify canonically $H^{n+1}_{\mathcal{M}}(Y^{(n)}_X, \mathbb{Q}(n+2))/H^1_{\mathcal{M}}(S, \mathbb{Q}(n+2))$ with $H^{n+1}_{\mathcal{M}}(Y^{(n)}_X, \mathbb{Q}(n+1)$; then we may define $Li_{mot} \in H^{n+1}_{\mathcal{M}}$, $\mathbb{Q}(n+1))$ to be a unique element that comes from $H^{n+2}_{\mathcal{M}}(Y^{(n+1)}_X, \mathbb{Q}(n+2))$ and maps to $1-t$ by β_n. For our aims this more precise definition is not necessary.

Let us compute the image of Π_{mot} by regulator maps. To do this note that in situation 1.1 we may compute the absolute cohomology of $Y^{(n)}_X$ using Leray spectral sequence for projection $\pi : Y^{(n)}_X \to X$. We may compute $R\pi_* \mathbb{Q}(n)_{Y^{(n)}_T}$ using the spectral sequence constructed as in 5.3. One gets immediately that $R^a\pi_*\mathbb{Q}(n)_{Y^{(n)}_T} = 0$ for $a \neq n$, and $R^n\pi_*\mathbb{Q}(n)_{Y^{(n)}_T}$ is mixed sheaf with $Gr^W_\cdot R^n\pi_* \mathbb{Q}(n)_{Y^{(n)}_T} = \mathbb{Q}(0)_T \oplus \ldots \oplus \mathbb{Q}(n)_T$. The differential d_1 in spectral sequence, just as in 5.3, equals to multiplication by $[t] \in H^1_{\mathcal{MT}}(X,\mathbb{Q}(1))$, and our sheaf splits over roots of unity (in particular, over 1) due to symmetries $\mu_a^{(n)}$ (see 5.3(ii), (iii)). Hence $R^n\pi_*\mathbb{Q}(n)_{Y^{(n)}_T}$ is just the sheaf $R_{\geq -2n}$ from 2.2. So the image of Π_{mot} by regulator map lies in $H^1_{\mathcal{MT}}(X, R(1)_{X \geq -2n+2})$. It coincides with corresponding Π from 2.1, 2.3, since it satisfies conditions of 2.1 (see 2.1.1)

5.6. Now let $\alpha \in F^* = T(F)$, $\alpha \neq 1$, be a root of unity. According to 5.3.3, we have a canonical decomposition $H^{n+1}_{\mathcal{M}}(Y^{(n)}_{\alpha}, \mathbb{Q}(n+1)) = \bigoplus_{1\leq k\leq n+1} H^1_{\mathcal{M}}(S,\mathbb{Q}(k))$. Let $Li_k(\alpha)_{mot} \in H^1_{\mathcal{M}}(S, \mathbb{Q}(i))$, $i = 1,\ldots,n$, be components of $\Pi_{mot\ \alpha} := \alpha^{*}\Pi_{mot}$. Call them motivic cyclotomic elements. According to 4.2, the regulator map transforms $Li_k(\alpha)_{mot}$ to element $Li_k(\alpha)$. This implies the conjecture ( ) from [BK].