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HIGHER REGULATORS OF MODULAR CURVES

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In this paper I will show that the $H_{\mathcal{M}}$-groups of any curve, uniformized by modular functions, contain a subgroup whose image under the regulator map is exactly the subgroup predicted by the conjectures of [1] §3 (cf. also [2] §8) about the values of L-functions.

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In the following I will use notations from the paper [2]. In particular if X is a scheme, then $H^j_{\mathcal{M}}(X,\mathbb{Q}(i))$ is the subspace of Quillen's group $K_{2i-j}(X)\otimes\mathbb{Q}$ on which Adams operators ψ^p act by multiplication by p^i.

1. THE STATEMENT OF MAIN RESULT.

1.1. Modular curves, preliminaries and notations. In the following V denotes the two-dimensional space $\mathbb{A}^{f2}$ over finite adeles $\mathbb{A}^f$, $G:GL_2(\mathbb{A}^f) = GL(V)$.

1.1.1. Fix an integer $N > 3$. Then $M_{(N)}$ denotes the moduli space over $\mathbb{Q}$ of elliptic curves with level N structure, and $\overline{M}_{(N)}$ the one of generalized elliptic curves - the smooth compactification of $M_{(N)}$; put $M^{\infty}_{(N)} := \overline{M}_{(N)} - M_{(N)}$ with reduced scheme structure. Let $\pi_{(N)}:X_{(N)} \to M_{(N)}$ be the universal curve and $\alpha_{(N)}:(\mathbb{Z}/N\mathbb{Z})^2 \to X_{(N)}(M_{(N)})$ be its level N structure. The scheme $\overline{M}(N)$ is a smooth projective absolutely irreducible curve over the cyclotomic field $\mathbb{Q}[\zeta_N]$. One has compatible left actions of the group $GL_2(\mathbb{Z}/N\mathbb{Z})$ on the schemes above; an element $g\in GL_2(\mathbb{Z}/N\mathbb{Z})$ acts on $\mathbb{Q}[\zeta_N]$ by $g^*(\zeta_N) = \zeta_N^{\det(g)}$. One has also the action of $(\mathbb{Z}/N\mathbb{Z})^2$ on the scheme $X_{(N)}$ by finite order point translations along the fibers of $\pi_{(N)}$, so in fact the semi-direct product $GL_2(\mathbb{Z}/N\mathbb{Z})\ltimes(\mathbb{Z}/N\mathbb{Z})^2$ acts on $X_{(N)}$.

If N_1, N_2 are integers s.t. $N_2 | N_1$, then there are compatible natural morphisms $\overset{(-)}{M}_{N_2} \to \overset{(-)}{M}_{N_2}$, $X_{N_1} \to X_{N_2}$ that commute with $GL_2(\mathbb{Z}/N_i\mathbb{Z})$- and $(\mathbb{Z}/N_i\mathbb{Z})^2$-action and $\alpha(N_i)$ via the reduction map $\mathbb{Z}/N_1\mathbb{Z} \to \mathbb{Z}/N_2\mathbb{Z}$. Put $\overset{(-)}{M} := \varprojlim \overset{(-)}{M}_{(N)}$, $X := \varprojlim X_{(N)}$. These are schemes over the cyclotomic field field $\mathbb{Q}[\zeta] := \cup\, \mathbb{Q}[\zeta_N]$, $\zeta_{N_1}^{N_1/N_2} = \zeta_{N_2}$. There is a canonical left G- and $G \ltimes V$-action on them s.t. $G(\hat{\mathbb{Z}})$ and $\hat{\mathbb{Z}}^2$ act via the limits of actions on finite levels.

More generally, we will need Kuga-Sato schemes $X^\ell \xrightarrow{\pi_\ell} M$ (ℓ is an integer ≥ 0). These are ℓ-fold fiber products of X over M: $X^\ell := X \times_M \dots \times X$; so $X^0 = M$, $X^1 = X$. There is an action of $G \ltimes V^\ell$ on X^ℓ s.t. for any compact open subgroup $K \subset G \ltimes V^\ell$ the factor scheme $K\backslash X^\ell$ has finite type over $\mathbb{Q}$ and $X^\ell = \varprojlim_K K\backslash X^\ell$. Since for any $K_1 \subset K_2$ the morphism $K_1\backslash X^\ell \to K_2\backslash X^\ell$ is finite, we have $H^\cdot_{\mathcal{M}}(X^\ell, \mathbb{Q}(*)) = \varinjlim_K H^\cdot_{\mathcal{M}}(K\backslash X^\ell, \mathbb{Q}(*))$; for small K's the space $H^\cdot_{\mathcal{M}}(K X^\ell, \mathbb{Q}(*))$ coincides with the space of K-invariant vectors in $H^\cdot_{\mathcal{M}}(X^\ell, \mathbb{Q}(*))$. If H is any contravariant functor on schemes of finite type over $\mathbb{Q}$ we put by definition $H(X^\ell) := \varinjlim H(K\backslash X^\ell)$; one has a natural action of $G \ltimes V^\ell$ on $H(X^\ell)$. Same notation for M^∞.

1.1.2. Consider the action of G on the component $[M]^1$ of degree 1 of the motive of M. Let $\mathcal{R}$ be a complete set of pairwise non-isomorphic weight 2 parabolic $\overline{\mathbb{Q}}$-representations of G. Then one has $[M]^1 = \bigoplus_{V \in \mathcal{R}} V \otimes M_V$ where $M_V := \mathrm{Hom}_G(V, M)$. This decomposition induces the corresponding decomposition of $H^1_{DR}(\overline{M}) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} \supset \Omega^1(\overline{M}) \otimes \overline{\mathbb{Q}}$, and $H^1_B(\overline{M}, \mathbb{Q}(i)) \otimes \overline{\mathbb{Q}}$. Let index V denote the V-component of the representation; we have $\Omega^1(\overline{M})_V = \mathcal{O} \otimes \Omega^1(M_V)$ and so on. The $\overline{\mathbb{Q}}$-spaces $\Omega^1(M_V)$ and $H^1_B(M_V, \mathbb{Q}(i))$ are 1-dimensional (multiplicity one theorem), and one has $L(M_V, S) = L(V^*, S)$. Here the right L-function is the one of Hecke-Jacquet-Langlands of the representation V^* dual to V; these L-functions are holomorphic and satisfy the functional equation. Note, that this functional equation implies that

$L(V,S)$ have simple zeros at non-positive integers; for $i \in \mathbb{Z}$, $i \leq 0$ put $\ell(V,i) := \frac{d}{dS} L(V,S)|_{S=i} \in (\overline{\mathbb{Q}} \otimes \mathbb{R})^*$.

1.2. <u>Construction of the elements in</u> $H^\cdot_{\mathcal{M}}(\overline{M})$. For any compact open subgroup $W \subset V^\ell$ the scheme $W\backslash X^\ell$ is smooth proper over M, so one has the Gysin map ${}_W\pi^\ell : H^a_{\mathcal{M}}(W\backslash X^\ell, \mathbb{Q}(b)) \to H^{a-2\ell}_{\mathcal{M}}(M, \mathbb{Q}(b-\ell))$. Clearly if $W_1 \subset W_2$, then ${}_{W_1}\pi^\ell_*$ coincides with $|W_1/W_2| \cdot \pi^\ell_{W_2 *}$ on $H^a_{\mathcal{M}}(W_2\backslash X^\ell \mathbb{Q}(b)) \subset H^a_{\mathcal{M}}(W_1\backslash X^\ell \mathbb{Q}(b))$. Let ν denote the 1-dimensional G-module, dual to the one of $\mathbb{Q}$-valued invariant measures on V; G acts on ν by the character $|\det|$. Clearly, we have a canonical $\nu^{\otimes \ell}$-valued measure μ on V^ℓ. The above shows that we have canonical G-map $\pi^\ell_* : H^a_{\mathcal{M}}(X^\ell, \mathbb{Q}(b)) \to H^{a-2\ell}_{\mathcal{M}}(M, \mathbb{Q}(b-\ell)) \otimes \nu^\ell$ that coincides with $\mu(W) \cdot {}_W\pi^\ell_*$ on $H^a_{\mathcal{M}}(W\backslash X^\ell, \mathbb{Q}(b))$.

Note that localization sequence together with the Borel theorem and [1] (cf. [2] no. 5) imply

<u>Lemma 1.2.1.</u> The restriction map $H^2_{\mathcal{M}}(\overline{M}, \mathbb{Q}(\ell+2)) \to H^2_{\mathcal{M}}(M, \mathbb{Q}(\ell+2))$ is injective for $\ell \geq 0$. One has $H^2_{\mathcal{M}}(\overline{M}, \mathbb{Q}(\ell+2)) = H^2_{\mathcal{M}}(\overline{M}, \mathbb{Q}(\ell+2))_{\mathbb{Z}}$ if $\ell > 0$. □

<u>Definition 1.2.2.</u> Put $H^2_{\mathcal{M}}(M, \mathbb{Q}(\ell+2))^{\text{parab}} := \pi^\ell_*(\{H^{\ell+1}_{\mathcal{M}}(X^\ell, \mathbb{Q}(\ell+1)), H^{\ell+1}_{\mathcal{M}}(X^\ell, \mathbb{Q}(\ell+1))\}) \otimes \nu^{-\ell}$. Clearly $H^2_{\mathcal{M}}(M, \mathbb{Q}(\ell+2))^{\text{parab}}$ is a $G(\mathbb{A}^f)$-submodule of $H^2_{\mathcal{M}}(M, \mathbb{Q}(\ell+2))$. The following theorem will be proved in 2.4.1 for $\ell > 0$ and in 5.2 for $\ell = 0$. Put $H^2_{\mathcal{M}}(\overline{M}, \mathbb{Q}(\ell+2))^{\text{parab}} := H^2_{\mathcal{M}}(M, \mathbb{Q}(\ell+2))^{\text{parab}} \cap H^2_{\mathcal{M}}(\overline{M}, \mathbb{Q}(\ell+2))$.

<u>Theorem 1.2.3.</u> If $\ell > 0$ then $H^2_{\mathcal{M}}(M, \mathbb{Q}(\ell+2))^{\text{parab}} = H^2_{\mathcal{M}}(\overline{M}, \mathbb{Q}(\ell+2))^{\text{parab}} \subset H^2_{\mathcal{M}}(\overline{M}, \mathbb{Q}(\ell+2))_{\mathbb{Z}}$. If $\ell = 0$ then $H^2_{\mathcal{M}}(\overline{M}, \mathbb{Q}(2))^{\text{parab}} \subset H^2_{\mathcal{M}}(\overline{M}, \mathbb{Q}(2))_{\mathbb{Z}}$.

1.3. <u>The main theorem.</u> Consider the subspace $P_\ell := r_{\mathcal{H}}(H^2_{\mathcal{M}}(\overline{M}, \mathbb{Q}(\ell+2))^{\text{parab}} \otimes \mathbb{Q} \subset H^1_B(\overline{M}, \mathbb{R}(\ell+1)) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$. This subspace is $G(\mathbb{A}^f)$-invariant, so one has $P_\ell = \bigoplus_{V \in \mathcal{R}} V \otimes P_{\ell V}$ for some $\overline{\mathbb{Q}}$ subspace $P_V \subset H^1_B(M_V, \mathbb{R}(\ell+1))$.

<u>Theorem 1.3.</u> One has $P_{\ell V} = \ell(V, -\ell) H^1_B(M_V, \mathbb{Q}(\ell+1))$. □

Clearly 1.2 and 1.1.3 imply that 1.3 is compatible with the conjectures of [1] §3, [2] §8. The conjectures themselves for the motives considered would be implied by 1.2, 1.3 and the unknown statement about the ranks of $H^{\cdot}_{\mathcal{M}}(M_N, \mathbb{Q}(*))$.

The proof of 1.3 goes as follows. First, in §2, we will see that the value of the regulator map r on an element $\pi^{\ell}_*(\{\alpha,\beta\})$ of $H^2_{\mathcal{M}}(M, \mathbb{Q}(\ell+2))^{\text{parab}}$ is a product of certain holomorphic and non-holomorphic Eisenstein series that correspond to residues of α, β at parabolic points (cf. 2.4). Then, in §3, we will prove that any reasonable function on parabolic points is the residue of some element from $H^{\ell+1}_{\mathcal{M}}(X^{\ell}, \mathbb{Q}(\ell+1))$ (for $\ell = 0$ this is just Manin-Drinfeld theorem). The results of §§2,3 describe explicitly the space P_{ℓ}. To find its V-components and their periods one has to compute Petersson's scalar products of elements of P_{ℓ} with parabolic weight two eigenforms of Hecke operators. This is what Rankin's method does (cf. §4). In §§2-4 we will suppose that $\ell > 0$; some minor changes needed to handle the case $\ell = 0$ are presented in §5 (this case was treated in [1] §5; we present it here for completeness sake).

2. REGULATORS AND EISENSTEIN SERIES.

We will use, throughout the text, the following notations. If Z is an analytic manifold over $\mathbb{R}$, dim Z = N, then $\mathcal{E}^{\cdot}_Z$, $C^{\cdot}_Z$ are complexes of $\mathbb{R}$-valued C^{∞}-class forms and currents respectively; one has $\mathcal{E}^{\cdot}_Z \subset C^{\cdot}_Z \otimes \mathbb{R}(-N)[-2N]$, $\mathcal{E}^n_Z \otimes \mathbb{C} = \bigoplus_{p+q=n} \mathcal{E}^{p,q}_Z$, $\mathcal{E}^{\cdot}_Z \otimes \mathbb{C} = \mathcal{E}^{\cdot}_Z \otimes \mathbb{R}(i) \oplus \mathcal{E}^{\cdot}_Z \otimes \mathbb{R}(i-1)$. Put $\mathcal{E}^{\cdot}(Z,\mathbb{R}(i)) = \Gamma(Z, \mathcal{E}^{\cdot}_Z \otimes \mathbb{R}(i)), \ldots$; for $\omega \in \mathcal{E}^{\cdot}(Z,\mathbb{C})$ let $\omega^{\overline{i}} \in \mathcal{E}^{\cdot}(Z,\mathbb{R}(i))$, $\omega^{(p,q)} \in \mathcal{E}^{p,q}(Z)$ denote its projections on corresponding spaces. If $\pi: Z \to T$ is a smooth map of relative dimension 1, then $\mathcal{E}^{\cdot}_{Z/T}, \ldots$ denote the sheaves on Z of relative forms along the fibers; we have a restriction to the fibers arrow $\#: \mathcal{E}^{p,q}_Z \to \mathcal{E}^{\ell,\ell}_{Z/T} \otimes \pi^* \mathcal{E}^{p-\ell,q-\ell}_T$. If π is proper, then we have the integration along the fibers map $\pi_*: C^{\cdot}(Z,\mathbb{R}(*)) \to C^{\cdot}(T,\mathbb{R}(*))$, $\mathcal{E}^{\cdot}(Z,\mathbb{R}(*)) \to \mathcal{E}^{\cdot-2\ell}(Z,\mathbb{R}(*-\ell))$ that factors through $\#$. For a bundle S over Z let $\mathcal{E}_{C^{\infty}}$ be the sheaf of its C^{∞}-class sections; so $\Omega^n_{C^{\infty}} = \mathcal{E}^{n,0}_Z$. If $Z_{\mathbb{Q}}$ is a scheme over $\mathbb{Q}$, put $\mathcal{E}^{\cdot}(Z) = \cdot(Z \otimes \mathbb{R}), \ldots$.

2.1. Preliminaries. In this n° we will recall some basic facts about residues and Eisenstein series. Let $\omega^{\ell} := (\omega^1)^{\otimes \ell} = \pi^{\ell}_*(\Omega^{\ell}(X^{\ell}/M))$ be the sheaf on M of weight ℓ modular forms.

2.1.0. Traces. Define the direct image map $\pi^{\ell}_*: \mathcal{E}^{\cdot}(X^{\ell}, \mathbb{R}(*)) = \mathcal{E}^{\cdot-2\ell}(M, \mathbb{R}(*-\ell)) \otimes \nu^{\ell}, \ldots$, to be $\mu(W) \cdot {}_W\pi^{\ell}_*$ on $\mathcal{E}^{\cdot}(W \backslash X^{\ell}, \mathbb{R}(*))$; same definition for currents and cohomology classes; these are well-defined G-maps. We have a canonical G-isomorphism $\omega^{\ell}_{C^{\infty}} \otimes \overline{\omega}^{\ell}_{C^{\infty}} \xrightarrow{\sim} \mathcal{O}_{C^{\infty}} \otimes \nu^{\ell}$, $\varphi \cdot \overline{\psi} := \pi^{\ell}_*(\varphi \wedge \overline{\psi})$. Define trace maps $\mathcal{E}^2(\overline{M}, \mathbb{R}(1)) \to \mathbb{R}, H^2_B(\overline{M}, \mathbb{Q}(1)) \to \mathbb{Q}$ to be $\mu'(K)$-times integration over the fundamental cycle of $K \backslash \overline{M}$ (here μ' is an invariant $\mathbb{Q}$-valued measure on G) for sufficiently small compact open subgroup $K \subset G$. These are also well-defined G-maps; they define G-pairings: Petersson's scalar product $(\ ,\ ): \mathcal{E}^{\ell,0}(\overline{M}) \otimes \overline{\mathcal{E}^{1,0}}(\overline{M}) \to \mathbb{R}, (\alpha,\beta) = \operatorname{Tr} \alpha \wedge \beta$ and Poincaré duality $H^1_B(\overline{M}, \mathbb{Q}(i)) \otimes H^1_B(\overline{M}, \mathbb{Q}(1-i)) \to \mathbb{Q}$.

2.1.1. Cusps. Recall the standard parametrization of M^{∞}. Let $\hat{X}$ be a group scheme over $\overline{M}$ obtained by adding to X the neutral components of fibers of Neron model over M^{∞}. More precisely, for a compact open subgroup $W \subset V$, one has the group scheme ${}_W\hat{X}$ of finite type over $\overline{M}$ s.t. for sufficiently small $K \subset G_W, \hat{X}$ coincides with inverse image of the scheme ${}_{KW}\hat{X}$ over $K \backslash \overline{M}$, where ${}_{KW}\hat{X}$ is $KW \backslash X$ with added neutral components of fibers of Neron model. When W varies the schemes ${}_W\hat{X}$ form an obvious projective system; put $\hat{X} = \varprojlim {}_W X$. Clearly the G-action on X prolongs to the one on $\hat{X}$.

Let $x \in M^{\infty}$ be a point. Note that the fiber of any ${}_W\hat{X}_x$ over x is isomorphic to a multiplicative group G_m, so we may consider the fiber $\hat{X}_x$ as a group up to isogeny, isomorphic to G_m. Call a parameter on $\hat{X}_x$ an isomorphism $\tilde{G}_m \xrightarrow{t} \hat{X}_x$ where $\tilde{G}_m$ is G_m in the category of groups up to isogeny; clearly parameters form $\mathbb{Q}^*$-torsor, since $\mathbb{Q}^* = \operatorname{Aut}(\tilde{G}_m)$. Now note that the group $F^*/\mathcal{O}^*_x$ (F = field of rational functions on M, $\mathcal{O}_x$ = local ring of $x \in \overline{M}$) is a 1-dimensional $\mathbb{Q}$-space, so the elements of $\mathcal{O} \backslash \{0\}/\mathcal{O}^*_x \subset F^*/\mathcal{O}^*_x$ - call them parameters at x - form $\mathbb{Q}^{*+}$-torsor. Define an enhanced point to be a triple (x,t,q) where x is a parabolic point, t is a parameter on $\hat{X}_x$ and q is a parameter at x. Clearly the space $\tilde{M}^{\infty}$ of enhanced points is a right $\mathbb{Q}^* \times \mathbb{Q}^{*+}$-torsor over M^{∞}; it is also obviously supplied with left G-action. The standard Tate curve defines the enhanced point $x_0 \in \widetilde{M}^\infty(\mathbb{Q}[\zeta])$. The stabilizator of this point in G is $U(\mathbb{A}^f)$ where $U = \{\begin{pmatrix}1&0\\ *&1\end{pmatrix}\} \subset GL_2$. The elements $(\alpha,\beta)\in \mathbb{Q}^*\times\mathbb{Q}^{*+}$ and $v \in \mathrm{Aut}\,\mathbb{Q}[\zeta] = \hat{\mathbb{Z}}^*$ act on x_0 by right multiplication by matrices $\begin{pmatrix}\alpha\beta&0\\0&\alpha\end{pmatrix}$ and $\begin{pmatrix}1&0\\0&v\end{pmatrix}$ respectively. Since G acts on $\widetilde{M}^\infty$ transitively this describes $\widetilde{M}^\infty$ and M^∞ completely. In particular the underlying space $|M^\infty|$ of M^∞ is $G/[B(\mathbb{Q})(\mathfrak{D}\cdot U)(\mathbb{A}^f)]$, where $B=\{\begin{pmatrix}*&0\\ *&*\end{pmatrix}\}$, $\mathfrak{D} = \{\begin{pmatrix}1&0\\0&*\end{pmatrix}\}$.

Fixing the $\mathbb{C}$-point of $\mathbb{Q}[\zeta]:\zeta_N = e^{\frac{2\pi i}{N}}$, one has $M^\infty(\mathbb{C}) = G/B(\mathbb{Q})^+$, where $B(\mathbb{Q})^+ := \{g\in B(\mathbb{Q}): \det g > 0\}$.

2.1.2. Residues. Let $A = \mathbb{Q}$ or $\mathbb{R}$. Define a G-morphism $\mathrm{Res}^\ell : H_B^{\ell+1}(X^\ell, A(\ell+1)) \to H_B^0(\widetilde{M}^\infty, A)$ as follows. Consider the G-scheme $\hat{X}^\ell := \hat{X}\times_{\overline{M}}\dots\times_{\overline{M}}\hat{X}$ (ℓ-fold fiber product). For compact open $W\subset V^\ell$ and sufficiently small $K\subset G$ one has the residue map ${}_{KW}\mathrm{Res}^\ell = \prod_{x\in K\backslash M^\infty} \mathrm{Res}^\ell_x : H_B^{\ell+1}(KW\backslash X^\ell, A(\ell+1)) \to H_B^\ell({}_{KW}\hat{X}^{\ell\infty}, A(\ell)) = \prod_{x\in K\backslash M^\infty} H^\ell_B(\hat{X}^\ell_x, A(\ell))$ the boundary map in exact sequence of pairs $({}_{KW}\hat X^\ell, KW\backslash X^\ell)$ here ${}_{KW}X^{\ell\infty} = \coprod_{x\in K\backslash M^\infty}\hat X^\ell_x$ is fiber of ${}_{KW}X^\ell$ over $K\backslash M^\infty$. This map depends on the choice of K in the following way: if $K'\subset K$, $x\in K'\backslash M^\infty$ and e is the ramification index of $\pi: K'/\overline{M}\to K\backslash\overline{M}$ at x, then one has $e\cdot\pi^*\,{}_{KW}\mathrm{Res}^\ell_{\pi(x)} = {}_{K'W}\mathrm{Res}^\ell_x\,\pi^*$. For an enhanced point $\tilde x = (x,t,q)$ define ${}_Ke(\tilde x)\in\mathbb{Q}^{*+}$ by $q^{{}_Ke(\tilde x)} = {}_Kq_x$ in $F^*/\mathcal{O}^*_x$, where ${}_Kq_x$ is natural parameter at $x\in K\backslash\overline{M}$. Then the arrows ${}_Ke(\tilde x)\,{}_{KW}\mathrm{Res}^\ell_x$ for different K's are compatible, so one has the map $\mathrm{Res}^{\ell 1}_{\tilde x} := \varinjlim_{K,W} {}_Ke(\tilde x)\,{}_{KW}\mathrm{Res}^\ell_x : H_B^{\ell+1}(X^\ell, A(\ell+1)) \to H^\ell_B(\hat X^\ell_K, A(\ell))$. Finally define $\mathrm{Res}^\ell_{\tilde x}: H_B^{\ell+1}(X^\ell,A(\ell+1))\to A$ to be $\mathrm{Res}^\ell_{\tilde x}$ composed with $H^\ell_B(\hat X^\ell_x, A(\ell)) \xrightarrow{q^{\ell *}} H^\ell_B(\widetilde{G}^\ell_m, A(\ell)) \cong A$; put $\mathrm{Res}^\ell := \prod_{\tilde x}\mathrm{Res}^\ell_{\tilde x}$.

In the same way one defines Res^ℓ in any other cohomology theory; e.g. we have a G-map $\mathrm{Res}^\ell_{\mathcal{M}}: H^{\ell+1}_{\mathcal{M}}(X^\ell,\mathbb{Q}(\ell+1))\to H^0_{\mathcal{M}}(\widetilde M^\infty,\mathbb{Q}) = H^0(\widetilde M^\infty,\mathbb{Q})$ (the only thing one has to remark for general cohomology is that the last arrow $H^\ell_?(\widetilde G^\ell_m, \mathbb{Q}(\ell))\to\mathbb{Q}$ in the definition of Res comes from canonical projection $H^\cdot_?(G^\ell_m\times S,\mathbb{Q}(*)) = H^\cdot_?(S,\mathbb{Q}(*))\otimes\Lambda^\cdot(t_1,\dots,t_\ell)\to H^{\cdot-\ell}_?(S,\mathbb{Q}(*-\ell))$,

$t_1\wedge\dots\wedge t_\ell\mapsto 1$, where $t_i = p_i^*(t)$, $t\in H^1(G_m,\mathbb{Q}(1))$ is the canonical element). Clearly we have commutative square of G maps

$$\begin{array}{ccc} H_B^{\ell+1}(X^\ell, A(\ell+1)) & \xrightarrow{\mathrm{Res}^\ell_B} & H^0_B(\widetilde M^\infty, A) \\ \uparrow r_B & & \uparrow \\ H_{\mathcal M}^{\ell+1}(X^\ell, \mathbb{Q}(\ell+1)) & \xrightarrow{\mathrm{Res}^\ell_{\mathcal M}} & H^0(\widetilde M^\infty, \mathbb{Q}) \end{array}$$

Clearly for $(\alpha,\beta)\in\mathbb{Q}^*\times\mathbb{Q}^{*+}$ we have $\mathrm{Res}^\ell_{\tilde x(\alpha,\beta)} = \alpha^\ell\beta^{-1}\mathrm{Res}^\ell_{\tilde x}$. So if we define $\mathcal{F}^\ell_A\subset H^0_B(\widetilde M^\infty, A)$, $\mathcal{F}^\ell\subset H^0(\widetilde M^\infty,\mathbb{Q})$ to be the subspace of elements φ, s.t. $\varphi(\tilde x(\alpha,\beta)) = \alpha^\ell\beta^{-1}\varphi(\tilde x)$, then the image of Res^ℓ lies in this subspace. According to the end of 2.1.1 we may identify $\mathcal F^\ell_A$ with the space of all locally-constant A-valued functions φ on G s.t. for any $c\in\mathbb{A}^f$, $a,d\in\mathbb{Q}$, $ad>0$ we have $\varphi(X\begin{pmatrix}a&0\\c&d\end{pmatrix}) = a^{-1}d^{\ell+1}\varphi(X)$, and $\varphi(X\begin{pmatrix}1&0\\0&-1\end{pmatrix}) = \varphi(X)$; $\mathcal F^\ell\subset\mathcal F^\ell_{\mathbb Q}$ is the subspace of right $\begin{pmatrix}1&0\\0&\hat{\mathbb Z}^*\end{pmatrix}$-invariant ones.

2.1.3. Eisenstein series. Let $H^\pm := \mathbb{P}^1_{\mathbb{R}an}\setminus\mathbb{P}^1(\mathbb{R})$ be half-planes considered as analytic manifold over $\mathbb{R}$ with standard coordinate z_0. The group $GL_2(\mathbb R)$ acts on $H^\pm$ from the right by formula $z_0\begin{pmatrix}a&b\\c&d\end{pmatrix} = \frac{az_0+c}{bz_0+d}$. The (pro-analytic) manifold $(M\otimes\mathbb R)_{an}$ may be identified in a standard way with $H^\pm\times G/GL_2(\mathbb Q)$; the canonical action of G on M coincides with obvious left-multiplication action. Also the semi-direct product $GL_2(\mathbb R)\ltimes\mathbb R^2$ acts on the product of $H^\pm$ and affine line $\mathbb A^1$ by formula $(z_0,z_1)(g,(a_1,a_2)) = (z_0 g, (bz_0+d)^{-1}z_1 + a_1 z_0 g + a_2)$ and we have $X\otimes\mathbb R_{an} = [(H^\pm\times\mathbb A^1)\times(G\times V)]/GL_2(\mathbb Q)\ltimes\mathbb Q^2$, $(X^\ell\otimes\mathbb R)_{an} = [(H^\pm\times\mathbb A^\ell)\times(G\times V^\ell)]/GL_2(\mathbb Q)\ltimes(\mathbb Q^2)^\ell$.

Put $\widetilde X^\ell := [(H^\pm\times A^\ell)\times(G\times V^\ell)]/B(\mathbb Q)\ltimes(\mathbb Q^2)^\ell$ where $B=\{\begin{pmatrix}*&0\\ *&*\end{pmatrix}\}\subset GL_2$, and let $p^\ell:\widetilde X^\ell\to(X^\ell\otimes\mathbb R)_{an}$ be the projection. These are etale maps. The connected components of $\widetilde X^\ell$ are in 1-1 correspondence with $M^\infty\otimes\mathbb R$: the map $p = p^0: \widetilde M := \widetilde X^0\to(M\otimes\mathbb R)_{an}$ identifies $\widetilde M$ with the disjoint union of some punctured neighborhoods of parabolic points; for example the standard point x_0 corresponds to the component $(H^+ \times 1)$.

The inverse image of ω^ℓ to $H^+ \times G$ has canonical trivialization given by section $\mathscr{X}^\ell := (2\pi i)^\ell dz_1 \wedge \ldots \wedge dz_\ell$, where z_i, $i \geq 1$ are coordinates on $\mathbb{A}^\ell$. Let ω be an $\ell+1$-form on X^ℓ, i.e., the section of the sheaf $\Omega^1 \otimes \omega^\ell$ on M. If $\omega \in \Omega^{\ell+1}(X^\ell \otimes \mathbb{R})$ has logarithmic singularities at infinity, then its inverse image to $H^+ \times G$ equals to $\omega(q)\cdot\frac{dq}{q} \wedge \mathscr{X}^\ell$, where $\omega(q) = \sum_{a\in\mathbb{Q}^*} f_{a,\omega}(g)q^a$. Here $\frac{dq}{q} := 2\pi i dz_0$, $q^a := e^{2\pi iaz}$ and f_a is certain $\mathbb{C}$-valued function on G. It is easy to see that $f_0(\omega)$ belongs to $\mathfrak{F}^\ell_{\mathbb{R}}$ and coincides with the residue of cohomology class of ω.

Now for any $\varphi \in \mathfrak{F}^\ell_{\mathbb{R}}$ consider the form $\varphi(g)\frac{dq}{q} \wedge \mathscr{X}^\ell$ on $H^+ \times G$. This form comes from the unique one, also denoted by $\varphi(\frac{dq}{q} \wedge \mathscr{X}^\ell)$ on $\widetilde{M}$. Put $E^\ell(\varphi) := p_*(\varphi(\frac{dq}{q} \wedge \mathscr{X}^\ell))$ (we suppose that $\ell > 0$); this is a well-defined section of $\Omega^1 \otimes \omega^\ell$ on $(M \otimes \mathbb{R})_{an}$, since for $\ell > 0$ the series in question converges absolutely. One knows that this $(\ell+1)$-form on X^ℓ has logarithmic singularities at infinity; we will denote the same way by $E^\ell(\varphi)$ its cohomology class. The map $E^\ell: \mathfrak{F}^\ell_{\mathbb{R}} \to H^{\ell+1}_{DR}(X^\ell \otimes \mathbb{R}) = H^{\ell+1}_B(X^\ell, \mathbb{C})$ commutes with G-action and (by Manin-Drinfeld theorem) one has $E^\ell(\mathfrak{F}^\ell_A) \subset H^{\ell+1}_B(X^\ell, A(\ell+1))$. More precisely, let Φ^ℓ_A denote the intersection of $H^{\ell+1}_B(X^\ell, A(\ell+1))$ with the $(\ell+1)$st-term of the Deligne Hodge filtration on $H_{DR}(X^\ell \otimes \mathbb{R})$. Then $\mathfrak{F}^\ell_A \underset{\text{Res }\ell}{\overset{E^\ell}{\rightleftarrows}} \Phi^\ell_A$ are mutually inverse G-isomorphisms.

2.2. <u>Eisenstein series in $\mathcal{H}$-cohomology</u>. Note that the canonical exact sequence of G-modules $0 \to H^\ell_B(X^\ell, \mathbb{R}(\ell)) \to H^{\ell+1}_{\mathcal{H}}(X^\ell, \mathbb{R}(\ell+1)) \to \Phi^\ell_{\mathbb{R}} \to 0$ has a unique splitting $S: \Phi^\ell_{\mathbb{R}} \to H^{\ell+1}_{\mathcal{H}}(X^\ell, \mathbb{R}(\ell+1))$, since G-modules $H^\ell_B(X^\ell, \mathbb{R}(\ell))$ and $\Phi^\ell_{\mathbb{R}}$ have different weights. Put $E^\ell_{\mathcal{H}} := S\circ E^\ell: \mathfrak{F}^\ell_{\mathbb{R}} \to H^{\ell+1}_{\mathcal{H}}(X^\ell, \mathbb{R}(\ell+1))$; clearly this is unique right-inverse to $\text{Res}^\ell_{\mathcal{H}}$. We will need an explicit formula for $E^\ell_{\mathcal{H}}$ in the explicit presentation of [2] (5.7.1) for $H^{\ell+1}_{\mathcal{H}}(X^\ell, \mathbb{R}(\ell+1))$.

Let $C_i \in \mathbb{Q}$, $i = 0,\ldots,\ell$, be the numbers defined by induction: $C_0 = (\ell+1)^{-1}$, $iC_{i-1} = (\ell-i+1)C_i$ for $i > 0$. Define the C^∞-class $\mathbb{R}(\ell)$-valued ℓ-form $\mathscr{X}^\ell_{\mathcal{H}}$ on $H^+ \times \mathbb{A}^\ell$ by formula

$$\mathscr{X}^\ell_{\mathcal{H}} = -(2\pi i)^\ell \sum_{i,k=0}^{\ell} 2\pi C_i y_k (\Sigma \overset{(-)}{dz_0} \wedge \ldots \wedge \widehat{\overset{(-)}{dz_k}} \wedge \ldots \wedge \overset{(-)}{dz_\ell})^{(i,\ell-i)}$$

Here y_k is imaginary part of z_k, $\overset{(-)}{dz_i}$ means either dz_i or its complex conjugate, the form under the brackets is the sum of all possible products of forms $\overset{(-)}{dz_i}$ with $\overset{(-)}{dz_k}$ missed.

For example $\mathscr{X}^1_{\mathcal{H}} = (4\pi^2 i)[-y_0(\overline{dz_1}-dz_1)+y_1(\overline{dz_0}-dz_0)]$. The direct computation shows that $d\mathscr{X}^\ell_{\mathcal{H}} = (\frac{dq}{q} \wedge \mathscr{X}^\ell)^{\overline{\ell}}$.

For $\varphi \in \mathfrak{F}^\ell_{\mathbb{R}}$ consider the unique C^∞-class ℓ-form on $\widetilde{X}^\ell$ such that its inverse image to $(H^+ \times \mathbb{A}^\ell) \times (G \times V^\ell)$ coincides with $\mathscr{X}^\ell_{\mathcal{H}}\cdot\varphi$; denote this form also by $\mathscr{X}^\ell_{\mathcal{H}}\cdot\varphi$. Define the form $E^\ell_{\mathcal{H}}(\varphi) \in \mathcal{E}^\ell(X, \mathbb{R}(\ell))$ by the formula $E^\ell_{\mathcal{H}}(\varphi) := p^\ell_*(\mathscr{X}^\ell_{\mathcal{H}}\cdot\varphi)$. Since the series converges absolutely, the definition is correct, and the above implies that $dE^\ell_{\mathcal{H}}(\varphi) = [E^\ell(\varphi)]^{\overline{\ell}}$. So $E^\ell_{\mathcal{H}}(\varphi)$ defines an element (of the same notation) in $H^{\ell+1}_{\mathcal{H}}(X^\ell, \mathbb{R}(\ell+1))$ (see [2] 5.7.1). Since the arrow defined by $E^\ell_{\mathcal{H}}: \mathfrak{F}^\ell_{\mathbb{R}} \to H^{\ell+1}_{\mathcal{H}}(X^\ell, \mathbb{R}(\ell+1))$ obviously commutes with G-action, it coincides with $E^\ell_{\mathcal{H}}$ from the beginning of the no.

2.3. <u>A product of Eisenstein series</u>. Denote by $\mathcal{E}^\ell(\varphi)$ the section of $\omega^\ell_{C^\infty}$ on M which is the $(\ell,0)$ component of the form $E^\ell_{\mathcal{H}}(\varphi)$ along the fibers of π; one has

$$\mathcal{E}^\ell(\varphi) = p_*(\varphi\cdot\frac{2\pi y_0}{\ell+1}\mathscr{X}) .$$

Let α be any $\ell+1$-form on X^ℓ with log-singularities at ∞. Consider $(1,0)$-form $\alpha\cdot\mathcal{E}^\ell(\varphi)$ on M (see 2.1.0). Since $\pi^1_*(dz_1 \wedge \overline{dz_1}) = \pi^{-1}y$, the projection formula shows that

$$\alpha\cdot\mathcal{E}^\ell(\varphi) = p_*(2^{2\ell+1}\cdot\pi^{\ell+1}\cdot(\ell+1)^{-1}\cdot y^{\ell+1}\varphi\cdot\alpha(q)\frac{dq}{q}).$$

Now consider a form $\pi^\ell_*(E^\ell_{\mathcal{H}}(\varphi_1) \cup E^\ell_{\mathcal{H}}(\varphi_2)) \in H^2_{\mathcal{H}}(M, \mathbb{R}(\ell+2)) = H^1_B(M, \mathbb{R}(\ell+1))$. According to [2] (5.7.1) one has

$\pi_*^\ell(E_{\mathcal{H}}^\ell(\varphi_1) \cup E_{\mathcal{H}}^\ell(\varphi_2)) = \overline{(E^\ell(\varphi_1)\cdot\bar{\mathcal{E}}^\ell(\varphi_2)-(-1)^\ell E^\ell(\varphi_2)\cdot\bar{\mathcal{E}}^\ell(\varphi_1))}^{\ell+1}$ - this is a closed 1-form on M.

Lemma 2.3.1. This form is the restriction to M of a certain closed current on $\overline{M}$.

Proof: Assume that we live on a certain $K\backslash\overline{M}$. Let x be a parabolic point, c_1, c_2 be the values of φ_1, φ_2 at x. Then, for a certain parameter q at x, one has $E^\ell(\varphi_1)\cdot\bar{\mathcal{E}}^\ell(\varphi_2) = (c_1\frac{dq}{q}+\mathcal{E}_1)(c_2(\log|q|)^{\ell+1}+f_2)$, where $\mathcal{E}_1$ is a 1-form holomorphic at x, and f_2 is a certain continuous function s.t. $f_2(x) = 0$. So $E^\ell(\varphi_1)\cdot\bar{\mathcal{E}}^\ell(\varphi_2) = c_1c_2(\log|q|)^{\ell+1}\cdot\frac{dq}{q}+u_{12}$, where u_{12} is an L^1-class form s.t. du is also of class L^1. This shows that $E^\ell(\varphi_1)\cdot\bar{\mathcal{E}}^\ell(\varphi_2)$ is restriction to $K\backslash M$ of the, same noted, current on $\overline{M}$ defined as $2c_1c_2(\ell+2)^{-1}(d(\log|q|)^{\ell+1})^{(1,0)}$ $+u_{12}$, where u_{12}, $(\log|q|)^{\ell+1}$ are L^1-class currents on $\overline{M}$. Now $(E^\ell(\varphi_1)\cdot\bar{\mathcal{E}}^\ell(\varphi_2)-(-1)^\ell E^\ell(\varphi_2)\bar{\mathcal{E}}^\ell(\varphi_1))^{\ell+1}$ is either $\overline{(u_{12}-u_{21})}^{1}$ if ℓ is even, or $2\cdot(\ell+2)^{-1}\cdot c_1c_2d(\log|q|)^{\ell+2}+(u_{12}+u_{21})$ if ℓ is odd. Clearly these are closed currents. □

Now consider $E_{\mathcal{H}}^\ell(\varphi_i)$ as ℓ-forms on X^ℓ. We have the function $[\varphi_1,\varphi_2] := \pi_*^\ell(E_{\mathcal{H}}^\ell(\varphi_1)\wedge E_{\mathcal{H}}^\ell(\varphi_2))$ on M. One has

$$d[\varphi_1,\varphi_2] = \overline{[E^\ell(\varphi_1)\cdot\bar{\mathcal{E}}^\ell(\varphi_2)+(-1)^\ell E^\ell(\varphi_2)\cdot\bar{\mathcal{E}}^\ell(\varphi_1)]}^{\ell} . \tag{2.3.2}$$

Moreover, clearly $[\varphi_1,\varphi_2]$ has asymptotic $c_1\cdot c_2\cdot(\log|q|)^{\ell+1}$ at a parabolic point; so it defines L^1-class current on $\overline{M}$ and 2.3.2 holds as equality between currents on $\overline{M}$. This fact implies the following lemma, that will be important for us in §4.

Lemma 2.3.3. For any holomorphic 1-form ω on $\overline{M}$ one has $(\omega,\pi_*^\ell(E_{\mathcal{H}}^\ell(\varphi_1) \cup E_{\mathcal{H}}^\ell(\varphi_2)) = (-1)^{\ell+1}(\omega,E^\ell(\varphi_1)\cdot\bar{\mathcal{E}}^\ell(\varphi_2))$.

Proof: If ω_1, ω_1 are (1,0)-forms, then $\bar{\omega}_1\wedge\omega_2$ $= 2\cdot(-1)^\ell\overline{\omega_1}^{\ell}\wedge\omega_2$ for any ℓ. So by 2.3.1 we have

$\omega\wedge\pi_*^\ell(E_{\mathcal{H}}^\ell(\varphi_1) \cup E_{\mathcal{H}}^\ell(\varphi_2)) = \omega\wedge\overline{(E^\ell(\varphi_1)\cdot\bar{\mathcal{E}}^\ell(\varphi_2)+(-1)^{\ell+1}E^\ell(\varphi_2)\cdot\bar{\mathcal{E}}^\ell(\varphi_1)}^{\ell+1}$

$= -\omega\wedge\overline{(E^\ell(\varphi_1)\cdot\bar{\mathcal{E}}^\ell(\varphi_2)+(-1)^{\ell+1}E^\ell(\varphi_2)\cdot\bar{\mathcal{E}}^\ell(\varphi_1))}^{\ell}$

$= -2\omega\wedge\overline{(E^\ell(\varphi_1)\cdot\bar{\mathcal{E}}^\ell(\varphi_2))}^{\ell}+\omega\wedge d[\varphi_1,\varphi_2] = (-1)^{\ell+1}\omega\wedge\overline{E^\ell(\varphi_1)}\cdot\mathcal{E}^\ell(\varphi_2)$

$+d(\omega\cdot[\varphi_1,\varphi_2])$. This proves the lemma. □

2.4. A formula for regulator. Note that since $\dim X^\ell = \ell+1$ and M is not compact, the Leray spectral sequence shows that one has a commutative diagram of isomorphisms

$$\begin{array}{ccc} H_B^{2\ell+1}(X^\ell,\mathbb{R}(2\ell+1)) & \longrightarrow & H_{\mathcal{H}}^{2\ell+2}(X^\ell,\mathbb{R}(2\ell+2)) \\ \downarrow{\scriptstyle\pi_*^\ell} & & \downarrow{\scriptstyle\pi_*^\ell} \\ H_B^1(M,\mathbb{R}(\ell+1))\otimes\nu^{\otimes\ell} & \longrightarrow & H_{\mathcal{H}}^2(M,\mathbb{R}(\ell+2))\otimes\nu^{\otimes\ell} \end{array}$$

Lemma 2.4.1. For $\psi_1,\psi_2 \in H_{\mathcal{H}}^{\ell+1}(X^\ell,\mathbb{R}(\ell+1))$ one has $\psi_1 \cup \psi_2 = E_{\mathcal{H}}^\ell\mathrm{Res}_{\mathcal{H}}^\ell\psi_1 \cup E_{\mathcal{H}}^\ell\mathrm{Res}_{\mathcal{H}}^\ell\psi_2$.

Proof: Since ψ and $E_{\mathcal{H}}^\ell\mathrm{Res}_{\mathcal{H}}^\ell\psi$ have the same residues it suffices to show that $\psi_1 \cup \psi_2$ depends only on residues at ψ_i, or, equivalently, that $\mathrm{Res}_{\mathcal{H}}^\ell\psi_1 = 0$ implies that $\psi_1 \cup \psi_2 = 0$. Since the map $\mathrm{Res}^\ell:\Phi_{\mathbb{R}}^\ell \to \mathfrak{F}_{\mathbb{R}}^\ell$ is injective (cf. the end of 2.1.3) and $\Phi_{\mathbb{R}}^\ell = \mathcal{E}(H_{\mathcal{H}}^{\ell+1}(X^\ell,\mathbb{R}(\ell+1)))$, the kernel of residue map on this cohomology group coincides with $\alpha(H_B^\ell(X^\ell,\mathbb{R}(\ell)))$. Since one has $\alpha(x) \cup y = \alpha(x \cup \mathcal{E}(y))$ (cf. [2] no 1), the needed fact would follow from the nullity of $\cup$-product pairing $\underline{H}^\ell(X^\ell,\mathbb{R})\otimes\Phi_{\mathbb{R}}^\ell$ $\to \underline{H}^{2\ell+1}(X^\ell,\mathbb{R})(\ell+1)$. The Leray spectral sequence of π^ℓ reduces to two-step filtration on $\underline{H}^\cdot(X^\ell)$, compatible with $\cup$-product: $\underline{H}^{\cdot(1)} \subset \underline{H}^\cdot(X^\ell)$, $\underline{H}^{\cdot(0)} := \underline{H}^\cdot(X^\ell)/\underline{H}^{\cdot(1)}$. Since any $\ell+1$-form being restricted to the fibers is zero, one has $\Phi^\ell \subset H^{\ell+1(1)}$ and so the $\cup$-product factors through $\underline{H}^{\ell(0)}\otimes\Phi_{\mathbb{R}}^\ell$ $\to \underline{H}^{2\ell+1}(X^\ell,\mathbb{R})(\ell+1)$. Now the Hodge structure on $H^\ell(X^\ell,\mathbb{R})^{(0)}$ is Tate's structure of weight ℓ, the Hodge structure on $\Phi_{\mathbb{R}}^\ell$ is the one of weight 0, and the Hodge structure on $\underline{H}^{2\ell+1}(X^\ell,\mathbb{R})(\ell+1) \xrightarrow[\sim]{\pi_*^\ell} \underline{H}^1(M,\mathbb{R})(1)$ has weights 0, -1. Since $\ell \neq 0$, the map $\underline{H}^{\ell(0)}\otimes\Phi^\ell \to \underline{H}^{2\ell+1}$ is zero. □

Now let us return to 1.2.

Theorem 2.4.2. a) One has $H_{\mathcal{M}}^2(M,\mathbb{Q}(\ell+2))^{\mathrm{parab}}$ $\subset H^2(\overline{M},\mathbb{Q}(\ell+2))_{\mathbb{Z}}$ (for $\ell > 0$).

b) The subspace $r_{\mathcal{H}}(H_{\mathcal{M}}^2(M,\mathbb{Q}(\ell+2))^{\mathrm{parab}}) \subset H_B^1(\overline{M},\mathbb{R}(\ell+1))$

is generated by elements $\pi_*^\ell(E_{\mathcal{H}}^\ell(\varphi_1) \cup E_{\mathcal{H}}^\ell(\varphi_2)) = (E^\ell(\varphi_1)\cdot\overline{\mathcal{E}}^\ell(\varphi_2) - (-1)^\ell E^\ell(\varphi_2)\cdot\overline{\mathcal{E}}^\ell(\varphi_1))^{\overline{\ell+1}}$, where the φ_i run through $\mathrm{Res}^\ell H_{\mathcal{M}}^{\ell+1}(X^\ell, \mathbb{Q}(\ell+1))$.

Proof: Clearly the functoriality of $r_{\mathcal{H}}$ together with 2.4.1 imply that $r_{\mathcal{H}}\pi_*^\ell(\{\alpha,\beta\}) = \pi_*^\ell(r_{\mathcal{H}}(\alpha) \cup r_{\mathcal{H}}(\beta)) = \pi_*^\ell(E_{\mathcal{H}}^\ell \mathrm{Res}_{\mathcal{M}}^\ell \alpha \cup E_{\mathcal{H}}^\ell \mathrm{Res}_{\mathcal{M}}^\ell \beta)$. To prove a) consider the commutative diagram

$$\begin{array}{ccccccc}
0 \to & H_{\mathcal{M}}^2(\overline{M}, \mathbb{Q}(\ell+2)) & \to & H_{\mathcal{M}}^2(M, \mathbb{Q}(\ell+2)) & \to & H_{\mathcal{M}}^1(M^\infty, \mathbb{Q}(\ell+1)) \\
 & \downarrow & & \downarrow & & \downarrow \\
0 \to & H_B^1(\overline{M}, \mathbb{R}(\ell+1)) & \to & H_B^1(M, \mathbb{R}(\ell+1)) & \to & H_B^0(M^\infty, \mathbb{R}(\ell))
\end{array}$$

whose rows are localization sequences and columns are regulator maps. Since the right vertical arrow is injective by Borel's theorem, a) follows from the fact that $\pi_*^\ell(E_{\mathcal{H}}^\ell(\varphi_1) \cup E_{\mathcal{H}}^\ell(\varphi_2)) \in H_B^1(\overline{M}, \mathbb{R}(\ell+1))$ (see 2.3.4), and Lemma 1.2. □

This theorem proves Theorem 1.2 in case $\ell > 0$.

3. EISENSTEIN SYMBOLS.

In this section we will show that residue maps $\mathrm{Res}_{\mathcal{M}}^\ell : H_{\mathcal{M}}^{\ell+1}(X^\ell, \mathbb{Q}(\ell+1)) \to \mathfrak{B}^\ell$ are surjective for $\ell > 0$ (the case $\ell = 0$ is just Manin-Drinfeld theorem of §5). To do this we will construct the Eisenstein map $E_{\mathcal{M}}^\ell : \mathfrak{B}^\ell \to H_{\mathcal{M}}^{\ell+1}(X^\ell, \mathbb{Q}(\ell+1))$ right inverse to Res^ℓ (and such that $r_{\mathcal{H}} E_{\mathcal{M}}^\ell = E_{\mathcal{H}}^\ell$).

3.1. Construction of Eisenstein symbols. For an abelian scheme A/S and $L \in \mathbb{Z}$ let $[L] \in \mathrm{End}\, A/S$ be the multiplication by L endomorphism, ${}_L A := \mathrm{Ker}[L]$ be the subscheme of order L points, and ${}_L U(A) := A \setminus {}_L A$.

Let $U \subset X$ denote the complement of all finite order points. More precisely, for compact open $W \subset V$ put ${}_W U := \bigcap_L {}_L U(W \setminus X)$: then $U := \varprojlim_W {}_W U \subset X$. Clearly the action of $G \ltimes V$ on X leaves U invariant, so we have the induced $G \ltimes V$-action on U. The scheme $\mathcal{U}$ is a projective limit of schemes ${}_L U(KW \setminus X^\ell)$ of finite type over $\mathbb{Q}$ under the family of affine morphisms (so one has $H_{\mathcal{M}}^\cdot(\mathcal{U}, \mathbb{Q}(*)) = \varinjlim H_{\mathcal{M}}^\cdot({}_L U(K\cdot W \setminus X^\ell), \mathbb{Q}(*))$.

Consider the "abelian" scheme $X^{\ell+1} := X \times_M \cdots \times X$ (a projective limit of abelian schemes under isogenies); let $p_i : X^{\ell+1} \to X$, $i = 0, \ldots, \ell$ be the i-th projection. Define the "abelian" subscheme $X^{\ell'} \subset X^{\ell+1}$ to be the kernel of $P := \sum_{0 \le i \le \ell} p_i : X^{\ell+1} \to X$, let $p_i' := p_i|_{X^{\ell'}}$ (clearly $X^{\ell'}$ is isomorphic to X^ℓ via the isomorphism $(p_1', \ldots, p_\ell')$.) Put $U^{\ell+1} := U \times_M \cdots \times U$, $U^{\ell'} := X^{\ell'} \cap U^\ell$. On the schemes $X^{\ell+1}$, $U^{\ell+1}$ the following groups act: G, the permutation group $\Sigma_{\ell+1}$ (permutations of p_i), and $V^{\ell+1}$. The first two of them leave $X^{\ell'}$ invariant; the stabilizator of $X^{\ell'}$ in $V^{\ell+1}$ is $V^{\ell'} := \mathrm{Ker}(P := \Sigma p_i : V^{\ell+1} \to V)$ in obvious notations. So one has the action of semi-direct product $(G \times \Sigma_{\ell+1}) \ltimes V^{\ell+1}$ on $X^{\ell+1}$, $U^{\ell+1}$, and the one of $(G \times \Sigma_{\ell+1}) \ltimes V^{\ell'}$ on $X^{\ell'}$ and $U^{\ell'}$.

Before going further, let me introduce some notations. If H is any group and $\mathcal{M}$ is an H-module, then $\mathcal{M}_H := H_0(H, \mathcal{M})$ denotes the maximal factor space on which H acts trivially. So if Φ is any $(G \times \Sigma_{\ell+1}) \ltimes V^{\ell'}$-module, then $\Phi_{V^{\ell'}}$ is naturally a $G \times \Sigma_{\ell+1}$-module. If Ψ is any $\mathbb{Q}^*$-module and $a \in \mathbb{Z}$, let Ψ_a be the maximal factor-module of weight a of Ψ, i.e., the maximal factor space s.t. an element $r \in \mathbb{Q}^*$ acts on Ψ_a by multiplication on r^{-a}. If Ψ is in fact G-module, then it is $\mathbb{Q}^*$-module by $\mathbb{Q}^* \subset \mathbb{A}^{f*} = $ Center G, and Ψ_a is naturally a G-module. If Δ is a $\Sigma_{\ell+1}$-module, let Δ_{sgn} be the component of Δ on which $\Sigma_{\ell+1}$ acts by character sgn (this is canonically direct summand of Δ). We will combine these notations for $(G \times \Sigma_{\ell+1}) \ltimes V^{\ell'}$-modules: e.g. such a module Φ defines G-module $\Phi_{a, sgn, V^{\ell'}}$.

The following basic result will be proved in 3.2.

Theorem 3.1.1. a) The group $V^{\ell'}$ acts on $H_{\mathcal{M}}^\cdot(X^{\ell'}, \mathbb{Q}(*))$ trivially.

b) G-modules $H_{\mathcal{M}}^\cdot(X^{\ell'}, \mathbb{Q}(*))_{sgn}$ have weight ℓ, i.e., $H_{\mathcal{M}}^\cdot(X^{\ell'}, \mathbb{Q}(*)) = H_{\mathcal{M}}^\cdot(X^{\ell'}, \mathbb{Q}(*))_{\ell, sgn}$.

c) Consider the restriction map $H_{\mathcal{M}}^\cdot(X^{\ell'}, \mathbb{Q}(*)) \to H_{\mathcal{M}}^\cdot(U^{\ell'}, \mathbb{Q}(*))$.

The induced G-map $H^{\cdot}_{\mathcal{M}}(X^{\ell'},\mathbb{Q}(*))_{sgn} \to H^{\cdot}_{\mathcal{M}}(U^{\ell'},\mathbb{Q}(*))_{\ell,sgn,V^{\ell'}}$ is an isomoprhism. □

Now consider the $G \ltimes V$-module $\mathcal{T} := H^1_{\mathcal{M}}(U,\mathbb{Q}(1)) = \mathcal{O}^*(U)\otimes\mathbb{Q}$. The space $\mathcal{T}^{\otimes\ell}$ is naturally a $(G\times\Sigma_{\ell+1})\ltimes V^{\ell+1}$-module: the group $\Sigma_{\ell+1}$ acts by $\sigma(f_0\otimes\ldots\otimes f_\ell) = \mathrm{sgn}(\sigma) f_{\sigma^{-1}(0)}\otimes\ldots\otimes f_{\sigma^{-1}(\ell)}$. Clearly the map $\widetilde{\langle\ \rangle}:\mathcal{T}^{\otimes\ell+1}\to H^{\ell+1}_{\mathcal{M}}(U^{\ell'},\mathbb{Q}(\ell+1))$ given by the formula $\langle \widetilde{f_0,\ldots,f_\ell}\rangle := \{p_0'^{*}(f_0),\ldots,p_\ell^{*}(f_\ell)\}$ is a $(G\times\Sigma_{\ell+1})\ltimes V^{\ell'}$-morphism. It defines a G-morphism $\langle\ \rangle:\mathcal{T}^{\otimes\ell+1}_{\ell\ sgn\ V^{\ell'}} \to H^{\ell+1}_{\mathcal{M}}(U^{\ell'},\mathbb{Q}(\ell+1))_{\ell,sgn,V^{\ell'}} \overset{3.1.1}{=} H^{\ell+1}_{\mathcal{M}}(X^{\ell'},\mathbb{Q}(\ell+1))_{sgn} \subset H^{\ell+1}_{\mathcal{M}}(X^{\ell'},\mathbb{Q}(\ell+1))$ which we call the Eisenstein symbol map. Our first aim is to compute explicitly the source of $\langle\ \rangle$.

Again some simple technical notations and remarks. For a commutative p-adic type group W denote by $\psi(W)$, the space of $\mathbb{Q}$-valued Schwartz-Bruhat functions on W with obvious W-module structure, and let $\Phi(W)\subset\psi(W)$ be the submodule of functions of invariant integral zero. More precisely, one has an integration map $\int:\psi(W)\to\nu_W$, where ν_W is the space dual to the (1-dimensional) space of $\mathbb{Q}$-valued invariant measures on W, and $\Phi(W) := \mathrm{Ker}\int$. We have the following easy

Lemma 3.1.2. a) The arrow $\int$ identifies $\psi(W)_W$ with ν_W. One has $\Phi(W)_W = 0$.

b) For $\ell > 0$, let $P:W^{\ell+1}\to W$ be the sum of projections. Put $W^{\ell'} := \mathrm{Ker}\, P$; one has $\nu_{W^{\ell'}} = \nu_{W^{\ell+1}}\otimes\nu_W^{-1} = \nu_W^{\otimes\ell}$. The integration along the fibers map $P_*:\psi(W)^{\otimes\ell+1} = \psi(W^{\ell+1}) \to \psi(W)\otimes\nu^{\otimes\ell}$ induces isomorphisms $\psi(W)^{\otimes\ell+1}_{W^{\ell'}} \cong \psi(W)\otimes\nu^{\otimes\ell}$, $\Phi(W)^{\otimes\ell+1}_{W^{\ell}} \cong \Phi(W)\otimes\nu_W^{\otimes\ell}$. □

We need the case $W = V$. The group G acts on $\nu = \nu_V$ by the character $|\det|$ (so ν has weight 2), and 3.1.2 implies

Corollary 3.1.3. There is a natural G-isomorphism $\Phi(V)^{\otimes\ell+1}_{V^{\ell'}} = \Phi(V)^{\otimes\ell+1}_{sgn,V^{\ell'}} \cong \Phi(V)\otimes\nu^{\otimes\ell}$. □

Since the divisors of degree zero supported on points of finite order of elliptic curves have finite order in Pic, the divisor map defines the short exact sequence $0\to\mathcal{T}_0\to\mathcal{T}\xrightarrow{\mathrm{div}}\Phi(V)\to 0$ of $G\ltimes V$-modules where $\mathcal{T}_0 := \mathcal{O}^*(M)\otimes\mathbb{Q}$. This defines a filtration of length $\ell+2$ on $\mathcal{T}^{\otimes\ell+1}$ whose smallest subspace (the 0-th graded factor) is $\mathcal{T}_0^{\otimes\ell+1}$ and smallest factor-space (the $\ell+2$-one) is $\Phi(V)^{\otimes\ell+1}$.

Lemma 3.1.4. The sequence $0\to\mathcal{T}_0^{\otimes\ell+1}\to\mathcal{T}^{\otimes\ell+1}_{V^{\ell'}}\to\Phi(V)^{\otimes\ell+1}_{V^{\ell'}}\to 0$ is exact.

Proof: Since the functor $?\mapsto ?_{V^{\ell'}}$ is exact, it suffices to show that if L^i is the i-th graded factor of the above filtration and $i\neq 0, \ell+2$, then $L^i_{V^{\ell'}} = 0$. But L^i is the direct sum of modules of type $\Phi(V)^{\otimes i}\otimes\mathcal{T}_0^{\otimes\ell+1-i}$, and $(\Phi(V)^{\otimes i}\otimes\mathcal{T}_0^{\otimes\ell+1-i})_{V^{\ell'}} = \Phi(V)^{\otimes i}_{V^{\ell'}}\otimes\mathcal{T}_0^{\otimes\ell+1-i}$. If $i\neq\ell+1$ this equals $\Phi(V)^{\otimes i}_{V^i}\otimes\mathcal{T}_0^{\otimes\ell+1-i}$; if $i\neq 0$ this is zero by 3.1.2a). □

Lemma 3.1.5. The sgn-component of $\widetilde{\langle\ \rangle}:\mathcal{T}^{\otimes\ell+1}\to H^{\ell+1}_{\mathcal{M}}(U^{\ell'},\mathbb{Q}(\ell+1))$ is zero on $\mathcal{T}_0^{\otimes\ell+1}$.

Proof: For $f_i\in\mathcal{T}_0$ one clearly has $\langle\widetilde{f_0,\ldots,f_\ell}\rangle = \pi^*\{f_0,\ldots,f_\ell\}\in\pi^* H^{\ell+1}_{\mathcal{M}}(M,\mathbb{Q}(\ell+1))\subset H^{\ell+1}_{\mathcal{M}}(U^{\ell'},\mathbb{Q}(\ell+1))$. But the elements of $\pi^* H^{\cdot}_{\mathcal{M}}(M)$ are obviously invariant under $\Sigma_{\ell+1}$-action. □

Clearly 3.1.4 and 3.1.5 imply

Corollary 3.1.6. The Eisenstein symbol $\langle\ \rangle$ factors (uniquely) through $\mathcal{T}^{\otimes\ell+1}_{\ell,sgn,V}\to\Phi(V)^{\otimes\ell+1}_{\ell,sgn,V}$. □

And now 3.1.3 together with 3.1.6 define the Eisenstein map $E^{\ell}_{\mathcal{M}}:(\Phi(V)\otimes\nu^{\otimes\ell})_{-\ell} = \Phi(V)_\ell\otimes\nu^{\otimes\ell}\to H^{\ell+1}_{\mathcal{M}}(X^{\ell'},\mathbb{Q}(\ell+1))$.

Finally recall the construction of horispherical isomorphism $\tau:\Phi(V)_{-\ell}\otimes\nu^{\otimes\ell}\cong\mathfrak{F}^\ell$. First for any $\mathbb{C}$-valued Schwartz-Bruhat function g on $\mathbb{A}^f$ let $L(g,s)$ denote the corresponding L-function (i.e., $L(g,s)$ is the analytic continuation of the series $\sum_{n\in\mathbb{Q}^*_+} g(n)\cdot n^{-s}$ that converges absolutely for $\mathrm{Re}\, s > 1$). One knows that if g is $\mathbb{Q}$-valued then $L(g,-\ell-1)\in\mathbb{Q}$, and

$L(\cdot,-\ell-1):\psi(A^f)_{-\ell-1} \to \mathbb{Q}$ is an isomorphism (weight $-\ell-1$) is taken with respect to the $\mathbb{Q}^{*+}$-action on $\psi(A^f)$ s.t. $(rg)(x) = g(r^{-1}x))$. For a function $f \in \Phi(V)$ consider $f_1 \in \Phi(A^f)$ defined by the formula $f_1(x) = \int f(x,y)dy$ (dy is standard invariant measure on A^f); for $g \in G$ put $\tilde{\tau}(f)(g) := L((g^{-1}f)_1,-\ell-1)$. Clearly $\tilde{\tau}(f)$ is $\mathbb{Q}$-valued function on G s.t. $\tilde{\tau}(f)(g(\begin{smallmatrix} a & 0 \\ * & d \end{smallmatrix})) = \tilde{\tau}(f)(g)\cdot a^{-\ell-1}\cdot|d|^{-1}$ for any $d \in A^{*f}$ and $a \in \mathbb{Q}^{*+}$. Define the desired map $\tau:\Phi(V)_{-\ell} \otimes V^{\otimes\ell} \to \mathfrak{F}^{\ell}$ by the formula $\tau(f) = \tilde{\tau}(f)\cdot|\det|^{-\ell}$. It is easy to see that τ is a G-isomorphism.

Put $E^{\ell} := \tilde{E}^{\ell} \circ \tau^{-1}:\mathfrak{F}^{\ell} \to H^{\ell+1}_{\mathcal{M}}(X^{\ell},\mathbb{Q}(\ell+1))$; here we identified $X^{\ell'}$ with X^{ℓ} via $(p_1',\ldots,p_{\ell}')$. We have the following theorem to be proved in 3.3.

Theorem 3.1.7. One has $\mathrm{Res}^{\ell}_{\mathcal{M}}E^{\ell}_{\mathcal{M}} = \mathrm{Id}_{\mathfrak{F}^{\ell}}$. □

Corollary 3.1.8. $\mathrm{Res}^{\ell}_{\mathcal{M}}$ is surjective. □

Clearly 3.1.7 implies that $r_{\mathcal{H}}E^{\ell}_{\mathcal{M}} = E^{\ell}_{\mathcal{H}}$ (see 2.2).

3.2. Proof of Theorem 3.1.1. First recall the standard motivic decompositions of elliptic curves. Let $p:E \to S$ be an elliptic curve over some scheme S, and $e:S \to E$ be zero section. Put $R := H^2_{\mathcal{M}}(E \times_S E,\mathbb{Q}(1)) = \mathrm{Pic}(E \times_S E) \otimes \mathbb{Q}$. Then multiplication of correspondences defines ring structure on R: if $\alpha,\beta \in R$ then $\alpha*\beta := P_{13*}(P_{12}^*(\alpha)\cdot P_{23}^*(\beta))$ where $P_{ij}:E \times_S E \times_S E \to E \times_S E$ is projection on (i,j)-pair. The unit 1 for $*$ is the class of diagonal. The transposition $E \times_S E \to E \times_S E$ defines an involution $\alpha \to \alpha^t$ of R: one has $(\alpha*\beta)^t = \beta^t*\alpha^t$. One has natural R-module structure on any $H^j_{\mathcal{M}}(E,\mathbb{Q}(i))$: if $\alpha \in R$, $\beta \in H^j(E,\mathbb{Q}(i))$, then $\alpha*\beta := p_{1*}(\{\alpha,p_2^*\beta\})$, where $p_i:E\times E \to E$ is i-th projection. If f is any endomorphisms of the scheme E over S, then its graph Γ_f is an element of R and one has $\Gamma_{f_1\circ f_2} = \Gamma_{f_1}*\Gamma_{f_2}$; if $\alpha \in H^j_{\mathcal{M}}(X,\mathbb{Q}(i))$, then $\Gamma_f*\alpha = f_*(\alpha)$, ${}^t\Gamma_f*\alpha = f^*(\alpha)$.

Lemma 3.2.1. If $a \in E(S)$ is a point of finite order on E, then the graph of translation by a is equivalent to 1 in R. So this translation acts trivially on any $H^j_{\mathcal{M}}(E,\mathbb{Q}(i))$. □

Now consider the elements $P_0 := E \times e$, $P_2 := e \times E = \Gamma_{e\circ p} = {}^tP_0$ and $P_1 := 1 - P_0 - P_2$ of R. Then for any $\alpha \in H^j_{\mathcal{M}}(E,\mathbb{Q}(i))$ one has $P_0*\alpha = P^*e^*(\alpha)$, $P_2*\alpha = e_*P_*(\alpha)$, $P_1*\alpha = \alpha-P^*e^*\alpha-e_*P_*\alpha$. One has the following well known

Lemma 3.2.2. The elements P_i are mutually orthogonal projectors that define spectral decomposition of elements $\Gamma_{[L]} \in R$. Namely one has $P_i^2 = P_i$, $P_iP_j = 0$ for $i \neq j$ and $\Gamma_{[L]}*P_i = P_i*\Gamma_{[L]} = L^{2-i}P_i$ for any $L \in \mathbb{Z}$. □

For any $i = 0,1,2$ put $H^{\cdot}_{\mathcal{M}}(E,\mathbb{Q}(*))^{(i)} := P_iH^{\cdot}_{\mathcal{M}}(E,\mathbb{Q}(*))$. By the lemma $H^{\cdot}_{\mathcal{M}}(E,\mathbb{Q}(*)) = \oplus H^{\cdot}_{\mathcal{M}}(E,\mathbb{Q}(*))^{(i)}$ and $[L]$ acts on $H^{\cdot}_{\mathcal{M}}(\ )^{(i)}$ by multiplication by L^i.

Remark 3.2.3. Consider the tower $E \xrightarrow{[N]} E$, $N \in \mathbb{Z}$, of isogenies; put $\tilde{E} := \varprojlim_{[N]} E$. Then, since any $[N]^*: H^{\cdot}_{\mathcal{M}}(E,\mathbb{Q}(*))^{(i)} \to H^{\cdot}_{\mathcal{M}}(E,\mathbb{Q}(*))^{(i)}$ is an isomorphism, one has $H^{\cdot}_{\mathcal{M}}(\tilde{E},\mathbb{Q}(*)) = H^{\cdot}_{\mathcal{M}}(E,\mathbb{Q}(*))$ and one has natural decomposition $H^{\cdot}_{\mathcal{M}}(\tilde{E},\mathbb{Q}(*)) = H^{\cdot}_{\mathcal{M}}(\tilde{E},\mathbb{Q}(*))^{(i)}$ spectral for operators $[L]^*$, $L \in \mathbb{Q}$. In 3.2.5-3.2.7 we will refer to such a scheme (e.g. to the scheme X over M) as an elliptic curve without any commentaries.

Consider the localization sequence $\to H^{j-2}_{\mathcal{M}}(S,\mathbb{Q}(i-1)) \to H^j_{\mathcal{M}}(E,\mathbb{Q}(i)) \to H^j_{\mathcal{M}}({}_1U,\mathbb{Q}(i))$ where ${}_1U := E\setminus e(S) \overset{j}{\hookrightarrow} E$. Since $P_*e_* = \mathrm{id}$ the arrow e_* is injective and j^* is surjective. More precisely, we have isomorphisms $e_*:H^{j-2}_{\mathcal{M}}(S,\mathbb{Q}(i-1)) \cong H^j_{\mathcal{M}}(E,\mathbb{Q}(i))^{(2)}$, $j^*:H^j_{\mathcal{M}}(E,\mathbb{Q}(i))^{(0)} \oplus H^j_{\mathcal{M}}(E,\mathbb{Q}(i))^{(1)} \to H^j_{\mathcal{M}}({}_1U,\mathbb{Q}(i))$.

Now suppose that $N \in \mathbb{Z}$ is invertible in $\mathcal{O}_S$ and we are given a level N structure on E. Then $[N]:E \to E$ is a $(\mathbb{Z}/N\mathbb{Z})^2$-Galois covering; consider the induced covering $[N]:{}_NU(E) \to {}_1U$. Note that for any G-Galois covering $f:X \to Y$ one has $f^*:H^{\cdot}_{\mathcal{M}}(Y,\mathbb{Q}(*)) \cong H^{\cdot}_{\mathcal{M}}(X,\mathbb{Q}(*))^G = H^{\cdot}_{\mathcal{M}}(X,\mathbb{Q}(*))_G$ since K-theory modulo torsion has etale descent. By 3.2.1 $(\mathbb{Z}/N\mathbb{Z})^2$ acts on $H^{\cdot}_{\mathcal{M}}(E,\mathbb{Q}(*))$ trivially and the commutative diagram

$$\begin{array}{ccc} H^{\cdot}_{\mathcal{M}}(E,\mathbb{Q}(*)) & \xrightarrow{j_N^*} & H^{\cdot}_{\mathcal{M}}({}_NU,\mathbb{Q}(*)) \\ \uparrow{\scriptstyle [N]^*} & & \uparrow \\ H^{\cdot}_{\mathcal{M}}(E,\mathbb{Q}(*)) & \xrightarrow{j^*} & H^{\cdot}_{\mathcal{M}}({}_1U,\mathbb{Q}(*)) \end{array}$$

implies:

Lemma 3.2.4. The kernel of the restriction map $j_N^* H^{\cdot}_{\mathcal{M}}(E,\mathbb{Q}(*)) \to H^{\cdot}_{\mathcal{M}}({}_N U,\mathbb{Q}(*))$ is $H^{\cdot}_{\mathcal{M}}(E,\mathbb{Q}(*))^{(2)}$, and j_N^* induces the isomorphism $H^{\cdot}_{\mathcal{M}}(E,\mathbb{Q}(*))^{(0)} \oplus H^{\cdot}_{\mathcal{M}}(E,\mathbb{Q}(*))^{(1)} \xrightarrow{\sim} H^{\cdot}_{\mathcal{M}}({}_N U,\mathbb{Q}(*))_{(\mathbb{Z}/N\mathbb{Z})^2}$. □

3.2.5. Let us begin the proof of 3.1.1. First suppose that $\ell = 1$. Then $X^{1'}$ is an elliptic curve over M (cf. 3.2.3), and so 3.2.1 implies 3.1.1 a); 3.2.2 implies 3.2.1 b) since the action of transposition $\sigma \in \Sigma_2$ coincides with $[-1]$; finally 3.2.4 implies 3.1.1 c).

3.2.6. To treat the case $\ell > 1$ note that the scheme X^{ℓ} is naturally an elliptic curve over $X^{\ell-1}$ via any of the projections $q^j := (p_0,\ldots,\hat{p}_j,\ldots,p_{\ell-1}): X^{\ell} \to X^{\ell-1}$. The finite order point translations for elliptic curve q_j correspond to the action of the elements of V^{ℓ} whose components are all zero but j-th one. By 3.2.1 these elements act trivially on $H^{\cdot}_{\mathcal{M}}(X^{\ell},\mathbb{Q}(*))$, and so does the whole group V^{ℓ}. It is easy to see that the projectors P^j_i on $H^{\cdot}_{\mathcal{M}}(X^{\ell},\mathbb{Q}(*))$, that correspond to different q^j by 3.2.2, mutually commute, and so we have decomposition $H^{\cdot}_{\mathcal{M}}(X^{\ell},\mathbb{Q}(*)) = \oplus H^{\cdot}_{\mathcal{M}}(X^{\ell},\mathbb{Q}(*))^{(i_0,\ldots,i_{\ell-1})}$ sum over all $(i_0,\ldots,i_{\ell-1})$ s.t. $0 \leq i_j \leq 2$. For any $\vec{L} = (L_0,\ldots,L_{\ell-1}) \in \mathbb{Q}^{\ell}$ define $[\vec{L}] \in \operatorname{End} X^{\ell}/M$ by the formula $[\vec{L}](X_0,\ldots,X_{\ell-1}) = ([L_0]X_0,\ldots,[L_{\ell-1}]X_{\ell-1})$, one has $[L]_{X^{\ell}} = [(L,\ldots,L)]$. Since $[\vec{L}] = \Pi[L_j]_j$ where $[L_j]_j$ is the $[L_j]$-endomorphism for elliptic curve q^j, the action of $[\vec{L}]^*$ on $H^{\cdot}_{\mathcal{M}}(X^{\ell},\mathbb{Q}(*))^{(i_0,\ldots,i_{\ell-1})}$ coincides with multiplication by $\Pi L_j^{i_j}$ and $H^{\cdot}_{\mathcal{M}}(X^{\ell},\mathbb{Q}(*))^{(i_0,\ldots,i_{\ell-1})}$ is determined uniquely by this property.

3.2.7. Now consider the isomorphism $I := (P'_0,\ldots,P'_{\ell-1}): X^{\ell'} \to X^{\ell}$. The action of $V^{\ell'}$ on $X^{\ell'}$ corresponds to the action of V^{ℓ} on X^{ℓ}, so the above proves 3.1.1 a). To prove 3.1.1 b) it suffices to show that $T := I_*(H^{\cdot}_{\mathcal{M}}(X^{\ell},\mathbb{Q}(*))_{sgn}) \subset K' := H^{\cdot}_{\mathcal{M}}(X^{\ell},\mathbb{Q}(*))^{(1,\ldots,1)}$. This follows from two facts:

a) Projection of T in $H^{\cdot}_{\mathcal{M}}(X^{\ell},\mathbb{Q}(*))^{(2,\cdot,\ldots,\cdot)}$ is zero and b) The $\Sigma_{\ell+1}$-subspace of $H^{\cdot}_{\mathcal{M}}(X^{\ell'},\mathbb{Q}(*))$, generated by $I^*H^{\cdot}_{\mathcal{M}}(X^{\ell},\mathbb{Q}(*))^{(0,\cdot,\ldots,\cdot)}$ has zero sgn-component. In fact, by Σ_{ℓ}-invariance, a) implies that $T \subset K'' := \oplus H^{\cdot}_{\mathcal{M}}(\ )^{(i_j)}$, sum over all (i_j) s.t. $i_j = 0,1$. In the same way, b) implies that $T \cap K''' = 0$, where $K''' := \oplus H^{\cdot}_{\mathcal{M}}(\ )^{(i_j)}$ sum over all (i_j) s.t. $i_j = 0,1$ and some i_j is 0. Note that $K'' = K' \oplus K'''$ and for any $L \in \mathbb{Q}$ the operator $[L]^*$ acts on K' by multiplication by L^{ℓ}, and has eigenvalues $< L^{\ell}$ on K'''. Since T is $[L]$-invariant, this implies that $T \subset K'$. q.e.d.

Now to prove a) and b) consider the map $s = q_0 \circ I = (p'_1,\ldots,p'_{\ell-2}): X^{\ell'} \to X^{\ell-1}$. This map is invariant under the transposition in $\Sigma_{\ell+1}$ that permutes p'_0 and p'_{ℓ}. This implies that $S_*H^{\cdot}_{\mathcal{M}}(X^{\ell'},\mathbb{Q}(*))_{sgn} = 0$ and that sgn-component of $\Sigma_{\ell+1}$-space, generated by $S^*H^{\cdot}_{\mathcal{M}}(X^{\ell-1},\mathbb{Q}(*))$ is zero. Since the projection from a) is $e_{0*}s_*$, and the subspace in b) is $S^*H^{\cdot}_{\mathcal{M}}(X^{\ell-1},\mathbb{Q}(*))$, these facts are proven.

3.2.8. It remains to prove 3.1.1 c). It suffices to show that $H^{\cdot}_{\mathcal{M}}(U^{\ell'},\mathbb{Q}(*))_{sgn,V^{\ell'}}$ decomposes under the action of operators $[L]^*$, $L \in \mathbb{Q}$ into the sum of eigenspaces with eigenvalues $L^{\ell}, L^{\ell+1},\ldots,L^{2\ell-1}$, and the L^{ℓ}-eigenspace is just $H^{\cdot}_{\mathcal{M}}(X^{\ell'},\mathbb{Q}(*))_{sgn}$ (in fact one may show that L^i-eigenspace is isomorphic to $H^{\cdot}_{\mathcal{M}}(X^{2\ell-i'},\mathbb{Q}(*))_{sgn}$). We will do this using the induction by ℓ; for $\ell = 1$ this was shown in 3.2.5. So suppose we know the fact for $\ell' < \ell$.

First note that $I: X^{\ell'} \xrightarrow{\sim} X^{\ell}$ identifies $U^{\ell'}$ with $U^{\ell} \cap P_{\ell}^{-1}(U)$, where $P^{\ell}: X^{\ell} \to X$ is the sum of projections. So the connected component $P_{\ell}^{-1}(e)$ of $U^{\ell} \setminus U^{\ell'}$ is $U^{\ell-1'}$; V^{ℓ} acts on the connected components transitively with stabilizator $V^{\ell'}$. The scheme $U^{\ell} \setminus U^{\ell'}$ is projective limit of schemes $W \setminus (U^{\ell} \setminus U^{\ell'})$ of finite type over M ($W \subset V^{\ell}$ is compact open). One has a canonical isomorphism $H^{\cdot}_{\mathcal{M}}(U^{\ell} \setminus U^{\ell'},\mathbb{Q}(*))_{V^{\ell-1'}} = H^{\cdot}_{\mathcal{M}}(U^{\ell-1'},\mathbb{Q}(*))_{V^{\ell-1'}} \otimes \psi(V)$ (cf. 3.1.2) and so canonical G-isomorphism $H^{\cdot}_{\mathcal{M}}(U^{\ell} \setminus U^{\ell'},\mathbb{Q}(*))_{V^{\ell}} = H^{\cdot}_{\mathcal{M}}(U^{\ell-1'},\mathbb{Q}(*))_{V^{\ell-1'}} \otimes \nu$.

Now consider the localization sequence of the pair $(U^{\ell},U^{\ell'})$. One has a corresponding exact sequence

$$\cdots \to H^{\cdot}_{\mathcal{H}}(U^{\ell},\mathbb{Q}(*))_{V^{\ell}} \to H^{\cdot}_{\mathcal{H}}(U^{\ell'},\mathbb{Q}(*))_{V^{\ell'}}$$
$$\to H^{\cdot-1}_{\mathcal{H}}(U^{\ell}\setminus U^{\ell'},\mathbb{Q}(*-1))_{V^{\ell}} = H^{\cdot-1}_{\mathcal{H}}(U^{\ell-1'},\mathbb{Q}(*-1))_{V^{\ell-1'}} \otimes \nu .$$

This is a sequence of Σ_{ℓ}-modules, so we get an exact sequence

$$\cdots \to H^{\cdot}_{\mathcal{H}}(U^{\ell},\mathbb{Q}(*))_{\overline{\mathrm{sgn}},V^{\ell}} \to H^{\cdot}_{\mathcal{H}}(U^{\ell'},\mathbb{Q}(*))_{\overline{\mathrm{sgn}},V^{\ell'}}$$
$$\overset{\delta}{\to} H^{\cdot-1}_{\mathcal{H}}(U^{\ell-1'},\mathbb{Q}(*-1))_{\overline{\mathrm{sgn}},V^{\ell-1'}} \otimes \nu$$

where $\overline{\mathrm{sgn}}$ is the sgn-characeter of Σ_{ℓ}. Using 3.2.4 the same way as in 3.2.6 one sees that $j^*:H^{\cdot}_{\mathcal{H}}(X^{\ell},\mathbb{Q}(*)) \to H^1_{\mathcal{H}}(U^{\ell},\mathbb{Q}(*))$ is epimorphic, so any element of Ker δ came from $X^{\ell'}$. Moreover j^* induces an isomorphism $\oplus H^{\cdot}_{\mathcal{H}}(X^{\ell},\mathbb{Q}(*))^{(i_j)} \to H^{\cdot}_{\mathcal{H}}(U^{\ell},\mathbb{Q}(*))$, sum over all (i_j) s.t. $i_j = 0,1$. In particular, $[L]^*$, $L \in \mathbb{Q}$, acts on $H^{\cdot}_{\mathcal{H}}(U^{\ell},\mathbb{Q}(*))$ with eigenvalues $1,L,\ldots,L^{\ell}$ and j^* is injective on $I_*H^{\cdot}_{\mathcal{H}}(X^{\ell'},\mathbb{Q}(*))_{\mathrm{sgn}}$ by 3.2.7. But the induction hypothesis says that $[L]^*$ acts on $H^{\cdot-1}_{\mathcal{H}}(U^{\ell-1'},\mathbb{Q}(*-1))_{\overline{\mathrm{sgn}},V^{\ell-1'}} \otimes \nu$ with eigenvalues $L^{\ell+1},\ldots,L^{2\ell-1}$. So the long exact sequence splits into short ones, $H^{\cdot}_{\mathcal{H}}(U^{\ell},\mathbb{Q}(*))_{\overline{\mathrm{sgn}},V^{\ell}} \to H^{\cdot}_{\mathcal{H}}(U^{\ell'},\mathbb{Q}(*))_{\overline{\mathrm{sgn}},V^{\ell'}}$ is injective, and so $H^{\cdot}_{\mathcal{H}}(X^{\ell'},\mathbb{Q}(*))_{\mathrm{sgn}} \to H^{\cdot}_{\mathcal{H}}(U^{\ell'},\mathbb{Q}(*))_{\mathrm{sgn},V^{\ell'}}$ is injective.

Since $H^{\cdot}_{\mathcal{H}}(X^{\ell'},\mathbb{Q}(*))_{\mathrm{sgn}} = H^{\cdot}_{\mathcal{H}}(U^{\ell},\mathbb{Q}(*))_{V^{\ell}} \cap H^{\cdot}_{\mathcal{H}}(U^{\ell'},\mathbb{Q}(*))_{\mathrm{sgn},V^{\ell}}$ (as sgn is exact functor) the eigenvalues of $[L]^*$ on $H^{\cdot}_{\mathcal{H}}(U^{\ell},\mathbb{Q}(*))_{\mathrm{sgn},V^{\ell}}/H^{\cdot}_{\mathcal{H}}(X^{\ell},\mathbb{Q}(*))_{\mathrm{sgn}}$ are $L^{\ell+1},\ldots,L^{2\ell-1}$. This together with 3.1.1 b) proves the induction hypothesis for ℓ.

3.3. <u>Proof of Theorem 3.1.7</u>. Let us reformulate a little the statement of the theorem using the Fourier transform instead of τ. For a Schwartz-Bruhat function α on $\mathbb{A}^f$ put $\hat{\alpha}(x) := \int \alpha(y)\overline{\psi}(xy)dy$, where $\psi:\mathbb{A}^f \to \mathbb{C}^*$ is the standard additive character: $\psi(y) = \exp(2\pi i y)$ for $y \in \mathbb{Q}$; for a function φ on $V := \mathbb{A}^{f2}$ put $\hat{\varphi}(x) := \int \varphi(y)\overline{\psi}(\langle x,y\rangle)dy$ where $\langle x,y\rangle := x_1y_2-y_1x_2$ and dy is the standard measure. Recall that the functional equation relates $L(\alpha,s)$ with $L(\hat{\alpha},1-s)$; we need the case $s = -\ell-1$: for any $\alpha \in \psi(\mathbb{A}^f)$ s.t. $\alpha(-x) = (-1)^{\ell}\alpha(x)$ one has

$L(\alpha,-\ell-1) = 2(-2\pi i)^{-\ell-2}(\ell+1)!L(\hat{\alpha},\ell+2)$; if $\alpha(-x) = (-1)^{\ell-1}\alpha(x)$ then $L(\alpha,-\ell-1) = 0$. Now consider G-map $\widetilde{E}^{\ell}:\Phi(V)_{(-\ell)} \otimes v^{\ell} \to \Phi^{\ell}$ s.t. for $(z,g) \in H^+ \times G$ and $\varphi \in \Phi(V)$ one has

$$\widetilde{E}^{\ell}(\varphi)(z,g)$$
$$= 2(-2\pi i)^{-\ell-2}(\ell+1)!\Big(\sum_{(n_1,n_2)\in\mathbb{Q}^2\setminus\{0\}} \frac{\hat{\varphi}({}^tg^{-1}(n_1,n_2))}{(n_1z+n_2)^{\ell+2}}\Big)\frac{dq}{q}\wedge\mathcal{H}^{\ell}$$

(cf. 2.1.3).

Clearly one has $\widetilde{E}^{\ell}(\varphi) = E^{\ell}(\tau\varphi)$, so to prove the theorem it suffices to show that $r_B\widetilde{E}^{\ell}_{\mathcal{H}}(\varphi) = \widetilde{E}^{\ell}(\varphi)$ for any $\varphi \in \Phi(V) \otimes v^{\otimes\ell}$. To do this we will compute, after some preliminaries in no. 3.3.1-3.3.2, the left hand side explicitly.

3.3.1. For notations cf. the beginning of §2. Let Z be a smooth variety and $i:\mathcal{D} = \cup\mathcal{D}_j \subset Z$ be a divisor. If a holomorphic form ω on $Z\setminus\mathcal{D}$ has log-singularities along $\mathcal{D}$ then ω is locally of class L^1 on Z. So ω defines L^1-class current ${}^c\omega$ or simply ω on Z. Note that this inclusion $\Omega^{\cdot}_Z(\log D) \hookrightarrow \mathcal{C}_Z \otimes \mathbb{C}[-2\dim Z]$ does not commute with differentials. For example, if $f \in \mathcal{O}^*(Z\setminus\mathcal{D})$, then $d({}^cd\log f) = \delta_{\mathrm{div}\, f}$ where for any divisor $\mathcal{D}'$ we put $\delta_{\mathcal{D}'}$:= fundamental class of $\mathcal{D}'$ considered as current on Z.

Now let $\pi:Z \to T$ be a smooth proper map of relative dimension n s.t. $\mathcal{D}$ is transversal to the fibers of π. If $\nu \in \mathcal{E}^{\cdot}(Z)$, $\omega \in \Omega^{\cdot}_Z(\log\mathcal{D})$ then the current $\pi_*(\omega\wedge\nu)$ is of class C^{∞} and we may compute its value at any point $t \in T$ integrating $(\omega\wedge\nu)^{\#}$ along $\pi^{-1}(t)$.

3.3.2. Let us return to the modular curve. We will write X instead of $X \underset{\mathbb{Q}[\zeta]}{\otimes} \mathbb{C}$ (where $\mathbb{Q}[\zeta] \hookrightarrow \mathbb{C}$, $\zeta_n \mapsto \exp(\frac{2\pi i}{n})$) and so on, for short. Clearly X/M considered as C^{∞}-class group variety over M, has a canonical integrable connection. In particular for any holomorphic 1-form $\nu \in \Omega^1(X/M) = \omega(M)$ one has canonical $\widetilde{\nu} \in \mathcal{E}^{(1,0)}(X)$ s.t. $\widetilde{\nu}^{\#} = \nu$. One has $(p_1+p_2)^*(\widetilde{\nu}) = p_1^*(\widetilde{\nu}) + p_2^*(\widetilde{\nu})$ (here $p_i:X \underset{M}{\times} X \to X$ are projections), and $\widetilde{\nu}$ is uniquely determined by this property.

From now on fix such ν non-zero at any point (we may work locally on M; e.g. we may pass to $H^+ \times G$). Put $\omega := \pi_*(\nu\wedge d\widetilde{\nu})$.

This is holomorphic 1-form on M (the Kodaira-Spencer transform of ν), non-zero at any point. So for any (1,0)-form φ on X one has $\varphi = \alpha\tilde{\nu} + \beta\omega$ for some functions $\alpha, \beta \in \mathcal{E}^0(X)$. Let $\mathcal{V} := \pi_*(\nu \wedge \bar{\nu}) \in \mathcal{E}^0(M)$ be the volume of the fibers function; one has $(\nu \wedge d\tilde{\nu})^\# = \mathcal{V}^{-1}\omega\cdot\nu\wedge\bar{\nu}$.

Consider the local system $\Gamma := R^1\pi_*(\mathbb{Q}(1))$ on M. Our form ν defines the embedding $\nu:\Gamma \hookrightarrow \mathcal{O}_M$, $\gamma \mapsto \nu(\gamma) := \pi_*(\nu\wedge\gamma)$ and $X = \mathbb{C}\times M/\Gamma$. Now note that Γ coincides with the local system of characters of the fibers of π, since for any elliptic curve E over $\mathbb{C}$, viewed as topological group, one has (characters of E) $:= \mathrm{Hom}(E, S^1) = H^1(E, \mathbb{Z}(1))$. For $\gamma \in \Gamma$ denote by χ_γ the corresponding character; the differential of χ_γ along the fibers is $\mathcal{V}^{-1}(\nu(\gamma)\bar{\nu} - \bar{\nu}(\gamma)\cdot\nu)\cdot\chi_\gamma$.

For any current $\alpha \in C(X)$ define the Γ^*-valued current $\hat{\alpha}$ on M - Fourier transform of α - by the formula $\hat{\alpha}(\gamma) = \pi_*(\alpha\bar{\chi}_\gamma)$. Also if φ is a function on X put $\hat{\varphi} := \mathcal{V}^{-1}\widehat{\varphi\cdot\nu\wedge\bar{\nu}}$. Clearly one has $\widehat{\frac{\partial}{\partial\nu}\varphi}(\gamma) = \mathcal{V}^{-1}\nu(\gamma)\hat{\varphi}(\gamma)$; if φ is continuous, then $e^*(\varphi) = \sum_{\gamma\in\Gamma}\hat{\varphi}(\gamma)$.

This Fourier transform is related with the one on V as follows. The functions on V are divisors on X supported on $X\setminus U$; so one h s the map $\delta:\psi(V) \to C^{-2}(X, \mathbb{R}(-1))$ (cf. 3.3.1). We have $\hat{\delta}_{\mathcal{E}} = \hat{\mathcal{E}}|_\Gamma$.

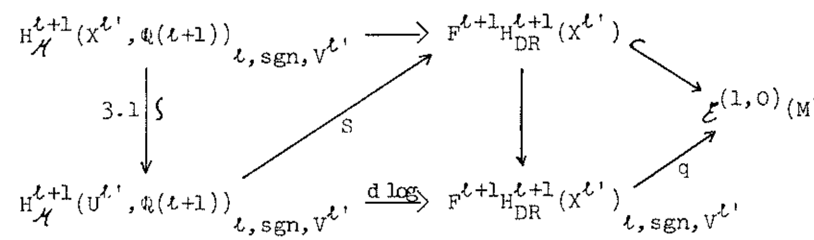
Now consider some $f \in \mathcal{O}^*(U)$; put $d\log f = \alpha\tilde{\nu} + \beta\omega$.

<u>Lemma 3.3.2.</u> One has $\hat{\alpha}(0) = 0$; if $\gamma \neq 0$, then $\hat{\alpha}(\gamma) = -\nu(\gamma)^{-1}\widehat{\mathrm{div}\, f}(\gamma)$, $\hat{\beta}(\gamma) = \nu(\gamma)^{-2}\widehat{\mathrm{div}\, f}(\gamma)$.

<u>Proof</u>: Since the current $d\log f + \overline{d\log f}$ is the differential of L^1-class current $2\log|f|$, one has $\hat{\alpha}(0) = \mathcal{V}^{-1}\pi_*(d\log f\wedge\bar{\nu}) = 2\mathcal{V}^{-1}\pi_*(d\log|f|\wedge\bar{\nu}) = 2\mathcal{V}^{-1}\pi_*(d(\log|f|\bar{\nu})) = 0$. Now we have $\delta^\#_{\mathrm{div}\, f} = d({}^c d\log f)^\# = (d\alpha\wedge\tilde{\nu})^\# = -\frac{\partial\alpha}{\partial\bar{\nu}}\nu\wedge\bar{\nu}$, and so $\widehat{\mathrm{div}\, f}(\gamma) = -\mathcal{V}\widehat{\frac{\partial\alpha}{\partial\nu}}(\gamma) = -\nu(\gamma)\hat{\alpha}(\gamma)$. To prove the formula for β, note that $0 = (\tilde{\nu}\wedge d(d\log f))^\# = (\alpha\tilde{\nu}\wedge d\tilde{\nu} + \tilde{\nu}\wedge d\beta\wedge\omega)^\# = (\alpha\mathcal{V}^{-1} + \frac{\partial\beta}{\partial\bar{\nu}})\omega\wedge\nu\wedge\bar{\nu}$. So $\frac{\partial\beta}{\partial\bar{\nu}} = -\alpha\mathcal{V}^{-1}$ and $\widehat{\frac{\partial\beta}{\partial\bar{\nu}}}(\gamma) = \mathcal{V}^{-1}\nu(\gamma)\hat{\beta}(\gamma) = -\mathcal{V}^{-1}\hat{\alpha}(\gamma) = \nu(\gamma)^{-2}\widehat{\mathrm{div}\, f}(\gamma)$. □

3.3.3. Now we may begin to prove Theorem 3.1.7. Put $\nu^{\ell'} := p_1'^*(\nu)\wedge\ldots\wedge p_\ell'^*(\nu) \in \Omega^\ell(X^{\ell'}/M)$ and consider the arrow

$q: F^{\ell+1}H^{\ell+1}_{DR}(U^{\ell'}) = \{\ell+1$-forms with log-singularities at ∞ on $U^{\ell'}\} \to \mathcal{E}^{(1,0)}(M)$, $q(\omega) = \pi_*^{\ell'}(\omega\wedge\bar{\nu}^{\ell'})$ (cf. 3.1.1). If $\omega \in F^{\ell+1}H^{\ell+1}_{DR}(X^{\ell'})$ then clearly one has $\omega = (-1)^{\frac{\ell(\ell-1)}{2}}\mathcal{V}^{-\ell}q(\omega)\wedge\nu^{\ell'}$. The arrow q obviously factors through $F^{\ell+1}H^{\ell+1}_{DR}(U^{\ell'})_{\ell,\mathrm{sgn},V^{\ell'}}$, and so the composition of q with $d\log = r_{DR}: H^{\ell+1}_{\mathcal{H}}(U^{\ell'}, \mathbb{Q}(\ell+1)) \to F^{\ell+1}H^{\ell+1}_{DR}(U^{\ell'})$ factors through $H^{\ell+1}_{\mathcal{H}}(X^{\ell'}, \mathbb{Q}(\ell+1))_{\ell,\mathrm{sgn},V^{\ell'}}$. The commutative diagram



implies that for any $\delta \in H^{\ell+1}_{\mathcal{H}}(U^{\ell'}, \mathbb{Q}(\ell+1))$ one has $s(\delta) = (-1)^{\frac{\ell(\ell-1)}{2}}\mathcal{V}^{-\ell}q(d\log\delta)\wedge\nu^{\ell'}$. For example, for any $f_0, \ldots, f_\ell \in \mathcal{O}^*(U)$ one has $d\log(\langle f_0, \ldots, f_\ell\rangle)$
$(-1)^{\frac{\ell(\ell-1)}{2}}\mathcal{V}^{-\ell}\pi_*^{\ell'}(d\log\widetilde{\langle f_0, \ldots, f_\ell\rangle}\wedge\bar{\nu}^{\ell'})\wedge\nu^{\ell'}$
$= (-1)^{\frac{\ell(\ell-1)}{2}}\mathcal{V}^{-\ell}\pi_*^{-\ell'}(\Lambda' d\log f_i\wedge\bar{\nu}^{\ell'})\wedge\nu^{\ell'}$, where $\Lambda' d\log f_i := p_0'^*d\log f_0\wedge\ldots\wedge p_\ell'^*d\log f_\ell$. To prove 3.1.7 it remains to compute this form in terms of $\mathrm{div}\, f_i$.

3.3.4. Consider the sum of projections map $P: X^{\ell+1} \to X$; $X^{\ell'} := \mathrm{Ker}\, P$. Put $\Lambda\, d\log f_i := p_0^*d\log f_0\wedge\ldots\wedge p_\ell^*d\log f_\ell \in \Omega^{\ell+1}(U^{\ell+1})$, $\tilde{\nu}^{\ell+1} := p_0^*(\tilde{\nu})\wedge\ldots\wedge p_\ell^*(\tilde{\nu}) \in \mathcal{E}^{(\ell+1,0)}(X^\ell)$, $\xi = \Lambda\, d\log f_i\wedge p^*(\tilde{\nu})\wedge\bar{\tilde{\nu}}^{\ell+1} \in \mathcal{E}^{(\ell+2,\ell+1)}(X^\ell)$. By 3.3.1, $\Lambda\, d\log f_i$ and ξ are currents on X^ℓ and $P_*(\xi) \in \mathcal{E}^{(2,1)}(X)$. The formula $P_*(\xi) = \eta\wedge\nu\wedge\bar{\nu}$ defines a certain form

$\eta \in \mathrm{Ker}(\Omega^{(1,0)}(X) \to \Omega^{(1,0)}(X/M))$. By 3.2.1 we have $\pi_*^{\ell'}(\Lambda\ d\log f_i \wedge \bar{\nu}^{\ell'}) = e^*(\eta)$ (where $e:M \to X$ is the zero section), and so, by 3.3.3, $d \log \langle f_0,\dots,f_\ell\rangle =$
$= (-1)^{\frac{\ell(\ell-1)}{2}} \nu^{-\ell} e^*(\eta) \wedge \nu^{\ell'}$.

3.3.5. Let us compute $e^*(\eta)$. Consider $P:X^{\ell+1} \to X$ as a map of topological tori. One has $\mathrm{Char}(X^{\ell+1}) = \Gamma^{\ell+1}$ and $P^*:\Gamma \to \Gamma^{\ell+1}$ is the diagonal map. Define the Fourier transform for currents and functions on $X^{\ell+1}$ the same way as in 3.3.2, using $(-1)^{\frac{\ell(\ell+1)}{2}} \nu^{-\ell-1}\nu^{\ell+1}\wedge\bar{\nu}^{\ell+1}$ instead of $\nu^{-1}\nu\wedge\bar{\nu}$.

Since $\eta \in \mathcal{E}^{(1,0)}(X)$, one has $e^*(\eta) = \sum_{\gamma\in\Gamma} \hat{\eta}(\gamma)$ $= \nu^{-1} \sum_{\gamma\in\Gamma} \widehat{P_*(\xi)}(\gamma) = \nu^{-1} \sum_{\gamma\in\Gamma} \hat{\xi}^{\#}(P^*(\gamma))$. Let α_i, β_i correspond to f_i as in Lemma 3.3.2; then
$\xi = (-1)^\ell \sum_{0\le i\le\ell} P_i^*(\beta_i)\, \Pi P_j^*(\alpha_j)\omega\wedge\nu^{\ell+1}\wedge\bar{\nu}^{\ell+1}$. We have
$\hat{\xi}^{\#}(\gamma_0,\dots,\gamma_\ell) = (-1)^{\frac{\ell(\ell+1)}{2}} \cdot \nu^{\ell+1} \sum_{0\le i\le\ell} \hat{\beta}_i(\gamma_i) \prod_{j\ne i} \hat{\alpha}_j(\gamma_j)\cdot\omega$, and so, by 3.3.2,

$$e^*(\eta) = (-1)^{\frac{\ell(\ell-1)}{2}} \nu^{\ell} \sum_{\gamma\in\Gamma} \sum_{0\le i\le\ell} \beta_i(\gamma) \prod_{j\ne i} \alpha_j(\gamma)\cdot\omega$$

$$= (-1)^{\frac{\ell(\ell+1)}{2}} \cdot \nu^{\ell}(\ell+1) \sum_{\gamma\in\Gamma} (\prod_i(\widehat{\mathrm{div}\ f_i})(\gamma)\cdot\nu(\gamma)^{-\ell-2})\cdot\omega.$$

Note that $\Pi(\widehat{\mathrm{div}\ f_i})(\gamma) = \widehat{P_*(\otimes\ \mathrm{div}\ f_i)}(\gamma)$; here $\otimes\ \mathrm{div}\ f_i \in \Phi(V^{\ell+1})$. We may combine this with 3.3.4 and the definition of $\tilde{E}_{\mathcal{H}}$ to see that for any $\varphi \in \Phi(V)$ one has $d\log \tilde{E}_{\mathcal{H}}(\varphi) = (-1)^\ell(\ell+1)(\sum'_{\gamma\in\Gamma} \hat{\varphi}(\gamma)\cdot\nu(\gamma)^{-\ell-2})\omega\wedge\nu^{\ell'}$.

This series coincides with $\tilde{E}^\ell(\varphi)$ from the beginning of 3.3. To see this, choose $\nu = dz_1$ on $H\times G$. Then $\omega = (-2\pi i)^{-1}dz_0$, and you get the desired formula.

4. CODA: THE VALUES OF L-FUNCTIONS.

In this section, we will prove Theorem 1.3 for $\ell > 1$.

4.1. Preliminaries on ϵ-factors, and periods. Suppose we are given two elements $a_1,a_2 \in \mathbb{C}\otimes_{\mathbb{Q}}\bar{\mathbb{Q}}$; say that a_1 is equivalent to a_2, $a_1\sim a_2$ if $a_1 \in \bar{\mathbb{Q}}^* a_2$. If U is any non-zero $\bar{\mathbb{Q}}$-vector space, then $a_1\sim a_2 \Longleftrightarrow a_1U = a_2U \subset \mathbb{C}\otimes_{\mathbb{Q}} U$.

Denote by $\mathcal{X}$ the group of $\bar{\mathbb{Q}}^*$-valued Dirichlet characters; for $\ell\in\mathbb{Z}$ denote by $\mathcal{X}^\ell \subset \mathcal{X}$ the set of characters of the same parity as ℓ. For $V \in \mathcal{R}$ (cf. 1.1.3) let $\theta(V):\hat{\mathbb{Z}}^* \to \bar{\mathbb{Q}}^*$ be its central character; one has $\theta(V) \in \mathcal{X}^0$. For $\chi\in\mathcal{X}$ and $V\in\mathcal{R}$ denote by $\chi\cdot V$ the twisted representation $\chi(\det)\otimes V$; one has $\theta(\chi\cdot V) = \chi^2\cdot\theta(V)$. Denote by $\epsilon(V,S), \epsilon(\chi,S) \in (\mathbb{C}\otimes\bar{\mathbb{Q}})^*$ the ϵ-factors in functional equations.

Lemma 4.1.1 ([3] (5.5)). Equivalence classes of $\epsilon(\chi,n), \epsilon(V,n), n \in \mathbb{Z}$, do not depend on n; denote them by $\epsilon(\chi)$, $\epsilon(V)$. One has $\epsilon(V)\sim\epsilon(\theta(V))$, $\epsilon(\chi_1\cdot\chi_2)\sim\epsilon(\chi_1)\cdot\epsilon(\chi_2)$.

Proof: For $a \in (\mathbb{C}\otimes\bar{\mathbb{Q}})^*$ define the function $\varphi_a:\mathrm{Aut}\ \mathbb{C} \to (\mathbb{C}\otimes\bar{\mathbb{Q}})^*$ by formula $\varphi_a(g) = g(a)a^{-1}$. Clearly one has $a_1\sim a_2 \Longleftrightarrow \varphi_{a_1} = \varphi_{a_2}$. So the lemma is implied by the fact that $\varphi_{\epsilon(\chi,n)}(g) = \chi(\sigma(g))$, $\varphi_{\epsilon(V,n)} = \theta(V)(\sigma(g))$ for $\chi\in\mathcal{X}^\ell$, $V\in\mathcal{R}$. Here $\sigma:\mathrm{Aut}\ \mathbb{C}\to\mathbb{Z}^*$ is the character of the action on the roots of unity. Decompose 's into the product of local ones: $\epsilon(\chi)\sim\sqrt{-1}^{\ell}\,\Pi\,\epsilon_p(\chi_p,\psi_p), \epsilon(V)\sim\Pi\epsilon_p(V_p,\psi_p)$. The above identities follow from the fact that $\epsilon_p(\chi_p,\psi_p(bx))$ $= \chi_p(b)\epsilon_p(\chi_p,\psi_p(x))$, $\epsilon_p(V_p,\psi_p(bx)) = \theta_p(V)(b)\epsilon_p(V_p,\psi_p(x))$ for $b\in\hat{\mathbb{Z}}_p^*$, since the values of ϵ_p-functions at integers are defined in a purely algebraical way and so $g\epsilon_p(?,\psi(x))$ $= \epsilon_p(?,\psi(\sigma_p(g)x))$. □

Since $L(\chi,1-\ell) \in \bar{\mathbb{Q}}^*$ for $\ell > 0$ and $\chi\in\mathcal{X}^\ell$; the functional equations imply

Lemma 4.1.2. a) For $\ell > 0$ and $\chi\in\mathcal{X}^\ell$ one has $L(\chi,\ell+2)\sim\epsilon(\chi)(2\pi i)^{\ell+2}$.

b) For $V\in\mathcal{R}$ and $\ell\ge 0$ one has $L(V,\ell+2)\sim\epsilon(V)\pi^{2\ell+2}\ell(V^*,-\ell)$ (cf. 1.1.3). □

For $V \in \mathcal{R}$ and $\ell \in \mathbb{Z}$ say that $a \in (\mathbb{C}\otimes\overline{\mathbb{Q}})^*$ is an ℓ-twisted period of V if $\Omega^1(M_V) = aH^1_B(M_V,\mathbb{Q}(-\ell))$ under the period isomorphism $\Omega^1(M_V)\otimes\mathbb{R} \simeq H^1_B(M_V,\mathbb{R}(-\ell))$. The equivalence class $\mathcal{L}_\ell(V)$ of such a's is well-defined.

Lemma 4.1.3. a) One has $L(V,1)\cdot H^1_B(M_V,\mathbb{Q}) \subset \Omega^1(M_V)$.

b) For any $V\in\mathcal{R}$ one may find even χ^+ and odd χ^- such that both $L(V\cdot\chi^\pm,1)$ are invertible; so in this case $L(V\chi^\pm,1)\sim\mathcal{L}_0(V\chi^\pm)$.

c) Let χ be the Dirichlet character of the same parity as $i\in\mathbb{Z}$. Then for any $V\in\mathcal{R}$ one has $\mathcal{L}_{ij}(V\cdot\chi)\sim(\pi\sqrt{-1})^i\epsilon(\chi)\mathcal{L}_j(V)$.

Proof: This lemma is well known: a) follows from Manin-Drinfeld, b) follows from surjectivity of Birch-Manin symbol map. To see c), decompose the zero component of M's motive $[\overline{M}]^0$ (= motive of cyclotomic field $\mathbb{Q}[\zeta]$) by the characters of $\mathrm{Aut}\,\mathbb{Q}[\zeta] = \hat{\mathbb{Z}}^* = \mathbb{A}^{f*}/\mathbb{Q}^{*+}$: $[\overline{M}]^0 = \oplus[\chi]$. The group $G(\mathbb{A}^f)$ acts on $[\chi]$ by $\chi^{-1}(\det)$. So the canonical pairing $[\overline{M}]^0\otimes[\overline{M}]^1 \to [\overline{M}]^1$ defines the isomorphism $[\chi]\otimes M_V \simeq M_{V\otimes\chi(\det)}$. If i has the same parity as χ, then $H^0_B([\chi],\mathbb{Q}(-i))$ and $H^0_{DR}([\chi])$ are 1-dimensional $\overline{\mathbb{Q}}$-spaces and one has $H^0_{DR}([\chi]) = \epsilon(\chi)\cdot(\pi\sqrt{-1})^i\cdot H^0_B([\chi],\mathbb{Q}(-i))$ (cf. [3] (6.5)). This, together with the (trivial) Kunneth formula proves c). □

Corollary 4.1.4. Let $V\in\mathcal{R}$ and $\ell\in\mathbb{Z}$. For any $\chi\in\mathcal{X}^\ell$ one has $L(\chi\cdot V,1)\in\epsilon(\chi)(\pi i)^{-\ell}\mathcal{L}_\ell(V)\cdot\overline{\mathbb{Q}}$. One may find $\chi\in\mathcal{X}^\ell$ s.t. $L(\chi\cdot V,1)$ is invertible. In this case $L(\chi\cdot V,1)\sim\epsilon(\chi)(\pi i)^{-\ell}\mathcal{L}_\ell(V)$. □

4.2. The use of Poincare duality. Let us reformulate 1.3. Denote by $\langle\ \rangle: H^1_B(\overline{M},\mathbb{Q}(-\ell))\otimes H^1_B(\overline{M},\mathbb{Q}(\ell+1))\to\mathbb{Q}$ the Poincare duality pairing: $\langle\alpha,\beta\rangle := \mathrm{Tr}(\alpha\smile\beta)$ (cf. 2.10). Consider our space $P_\ell = \bigoplus_{V\in\mathcal{R}} V\otimes P_{\ell V}\subset H^1_B(\overline{M},\mathbb{R}(\ell+1))\otimes\overline{\mathbb{Q}}$. Since $\dim H^1_B(M_V,\mathbb{Q}(\ell+1)) = 1$ one has

$$P_{\ell V^*} = \langle P_\ell, H^1_B(\overline{M},\mathbb{Q}(-\ell))_V\rangle H^1_B(M_{V^*},\mathbb{Q}(\ell+1))$$

$$= \langle P_\ell,\Omega^1(\overline{M})_V\rangle\mathcal{L}_\ell(V)^{-1}H^1_B(M_{V^*},\mathbb{Q}(\ell+1)) .$$

By 3.1, 2.4.2 and 2.3.3, we know that $\langle P_\ell,\Omega^1(\overline{M})_V\rangle\subset\mathbb{C}\otimes\overline{\mathbb{Q}}$ is the space generated by Petersson scalar products $(\omega, E^\ell(\varphi_1)\cdot\mathcal{E}^\ell(\varphi_2))$, where $\omega\in\Omega^1(\overline{M})_V$ and $\varphi_i\in\mathcal{F}^\ell$. Note that $\mathcal{F}^\ell\otimes\overline{\mathbb{Q}} = \bigoplus_{\chi\in\mathcal{X}^\ell} E_\chi$, where E_χ is an irreducible representation s.t. $L(E_\chi,s) = L(\chi,s-\ell-1)\cdot L(1,s)$. So 4.1.1, 4.1.2 and 4.1.4 imply that 1.3 follows from

(4.2.1) For any irreducible $V\subset\Omega^1(\overline{M})\otimes\overline{\mathbb{Q}}$ and $\chi\in\mathcal{X}^\ell$ the $\overline{\mathbb{Q}}$-space generated by scalar products $(\omega,E^\ell(e)\cdot\mathcal{E}^\ell(\varphi))\in\mathbb{C}\otimes\overline{\mathbb{Q}}$, $\omega\in V, e\in E_\chi$, $\varphi\in\mathcal{F}^\ell$, coincides with $\frac{L(V\cdot\chi,1)\cdot L(V,\ell+2)}{L(\chi\cdot\theta(V),\ell+2)}\cdot\overline{\mathbb{Q}}$.

This statement follows immediately from Rankin's trick. In the next section I will recall briefly the basic points we need; for details see e.g. [4].

4.3. Rankin's trick. First we need some facts about q-expansions and Mellin transforms. Let $\psi:\mathbb{A}^f\to\mathbb{C}^*$ be the character of $\mathbb{A}^f$ s.t. $\psi(t)=\exp(-2\pi i t)$ for $t\in\mathbb{Q}$, let $\psi_p:\mathbb{Q}_p\to\mathbb{C}^*$ be a local component of ψ. A Whittaker (or simply W-) function on G is a continuous $\mathbb{C}$- or $\mathbb{C}\otimes\overline{\mathbb{Q}}$-valued function f on G s.t. $f(g\begin{pmatrix}1&0\\u&1\end{pmatrix}) = \psi(u)f(g)$ for any $u\in\mathbb{A}^f$; similarly one defines W-functions on $G_p := GL_2(\mathbb{Q}_p)$. Say that a W-function on G is rational if for any $\sigma\in\mathrm{Aut}\,\mathbb{C}$ one has $\sigma(f(g)) = f(g\begin{pmatrix}1&0\\0&\theta(\sigma)^{-1}\end{pmatrix})$; and the same for W-functions on G_p.

Let ω be a $\ell+1$-form on $X^\ell\otimes\mathbb{C}$ with log-singularities at ∞. Its inverse image to $H^+\times G$ is $\omega_{(q)}\frac{dq}{q}\wedge\mathcal{X}^\ell$, where $\omega_{(q)} = \sum_{\alpha\in\mathbb{Q},\alpha\geq 0} f_{\alpha,\omega}(g)\cdot q^\alpha$ (see 2.1.3). Put $W(\omega) := f_{1,\omega}$. Then $W(\omega)$ is a W-function; if $\omega\in\Omega^{\ell+1}(X^\ell)$, then $W(\omega)$ is rational; if ω is parabolic, then

(4.3.1) $$\omega_{(q)}\frac{dq}{q}\wedge\mathcal{X}^\ell = \sum_{\alpha\in\mathbb{Q},\alpha>0}[W(\omega)dq\wedge\mathcal{X}^\ell]\begin{pmatrix}1&0\\0&\alpha\end{pmatrix} .$$

Let $E = \otimes E_p\subset\Omega^{\ell+1}(X^\ell,\log\infty)\otimes\overline{\mathbb{Q}}$ be an irreducible $\overline{\mathbb{Q}}$-representation. Then $W(E) = \otimes W(E_p)$, where $W(E_p)$ are certain spaces of rational $\mathbb{C}\otimes\overline{\mathbb{Q}}$-valued W-functions on G_p; this means that $W(E)$ is the space of linear combinations of functions $f(g) = \prod f_p(g_p)$ where $f_p\in W(E_p)$ and for almost all p, f_p is the spherical function, i.e., (unique) $GL_2(\mathbb{Z}_p)$-invariant function s.t. $f_p(1) = 1$. For $e \in E$ consider the function $K(e)$ on $\mathbb{A}^{f*}$ given by the formula $K(e)(a) := W(e)(\begin{pmatrix}1&0\\0&a^{-1}\end{pmatrix})$; this function is compactly supported in $\mathbb{A}^f$ and rational in the sense that $\sigma K(e)(a) = K(e)(a\cdot\theta(\sigma))$ for $\sigma \in \mathrm{Aut}\,\mathbb{C}$. Similarly one defines rational functions $K(e_p)$ on $\mathbb{Q}_p^*$ for $e_p \in E_p$. Put

$$L(e_p,s) := \int_{\mathbb{Q}_p^*} K(e_p)(a)|a|_p^{s-\ell-1}\, d^*a$$

$$= \Sigma\Big(\int_{\mathbb{Z}_p^*} K(e_p)(p^n a)d^*a\Big)\cdot p^{-n(s-\ell-1)}$$

(here d^*a is the standard invariant measure on $\mathbb{Q}_p^*$). This formal series is a rational function of parameter p^{-s} and the set $\{L(e_p,s), e_p\in E_p\}$ coincides with $\overline{\mathbb{Q}}[p^{-s},p^s]\cdot L(E_p,s)$, where $L(E_p,s)$ is the L-factor of E_p. The Euler product $L(E,s) = \Pi L(E_p,s)$ converges for $\mathrm{Re}\, s > \frac{\ell+3}{2}$; it prolongs holomorphically to any s and satisfies the functional equation for $s \leftrightarrow \ell+q-s$.

Now let $V = \otimes V_p \subset \Omega^1(\overline{M})\otimes\overline{\mathbb{Q}}$ be an irreducible parabolic representation; we may apply the above to it. For $\omega_p \in V_p$, $e_p \in E_p$ put $[\omega_p,e_p] := \int_{\mathbb{Q}_p^*} K(\omega_p)\cdot K(e_p)\cdot|a|^{s-\ell-2}d^*a$; again this series is a rational function of the parameter p^{-s}, and for a certain L-factor $L(V_p,E_p,s)$ with coefficients in $\overline{\mathbb{Q}}$, the $\overline{\mathbb{Q}}$-space, generated by $[\omega_p,e_p]_s$, $\omega_p\in V_p$, $e_p \in E_p$, is $\overline{\mathbb{Q}}[p^{-s},p^s]\cdot L(V_p,E_p,s)$. If both ω_p and e_p are spherical functions, then $[\omega_p, {}_p]_s = L(V_p,E_p,s)\cdot L(\theta(V_p)\cdot\theta(E_p),-\ell-2)^{-1}$. The Euler product $L(V,E,s) = \Pi L(V_p,E_p,s)$ converges for $\mathrm{Re}\, s > \ell+3$ and prolonges holomorphically to any s (if $\ell = 0$ this is valid for $E \neq V^*$). Consider the $\overline{\mathbb{Q}}$-linear function $[\ ,\ ]_s : V\otimes_{\mathbb{Q}} E \to \mathbb{C}\otimes\overline{\mathbb{Q}}$ defined by $[\otimes\omega_p,\otimes e_p]_s = \Pi[\omega_p,e_p]_s$ for $\mathrm{Re}\, s>\ell+3$; for arbitrary s this function is meromorphic, it is holomorphic for $\mathrm{Re}\, s \geq \frac{\ell+3}{2}$. Since local factors $L(\theta(V_p)\cdot\theta(E_p),2s-\ell-2)^{-1}$ have no zeros for $\mathrm{Re}\, s\geq\frac{\ell+2}{2}$ and take values in $\overline{\mathbb{Q}}$ at integers, the above shows that

(4.3.2) for $n\in\mathbb{Z}$, $n\geq\frac{\ell+3}{2}$ the $\overline{\mathbb{Q}}$-space generated by $[\omega,e]_n$, $\omega\in V$, $e\in E$, is $L(V,E,n)\cdot L(\theta(V)\cdot\theta(E),2n-\ell-2)^{-1}\cdot\overline{\mathbb{Q}}$.

Let us reformulate (4.3.2) a bit. Denote by $\mathcal{B}_{s,\ell}$ the space of $\mathbb{C}\otimes\overline{\mathbb{Q}}$-valued continuous functions φ on G s.t. $\varphi(g\cdot\begin{pmatrix}a&0\\c&d\end{pmatrix}) = \varphi(g)\cdot|a/d|_f^s\cdot a^\ell$ for any $a\in\mathbb{Q}^*$, $c\in\mathbb{A}^f$, $d\in\mathbb{A}^{f*}$; if s is integral let $\mathcal{B}^0_{s,\ell}\subset\mathcal{B}_{s,\ell}$ be the subspace of $\overline{\mathbb{Q}}$-valued ones; clearly $\mathcal{B}^0_{\ell+1,\ell} = \mathcal{F}^\ell\otimes\overline{\mathbb{Q}}$. Fix a non-zero invariant measure $d\mu$ on $G/\mathcal{D}(\mathbb{Z})\mathcal{U}(\mathbb{A}^f)$, then for $\varphi\in\mathcal{B}_{\ell+1,2\ell}$ the measure $\varphi\cdot|\det|_f^\ell\cdot d\mu$ is right $B(\mathbb{Q})$-invariant, so we have non-degenerate G-pairing $(\ ,\ )\cdot\mathcal{B}_{s,\ell}\otimes\mathcal{B}_{\ell+1-s,\ell}\otimes\nu^\ell\to\mathbb{C}\otimes\overline{\mathbb{Q}}$, $\mathcal{B}^0_{n,\ell}\otimes\mathcal{B}^0_{\ell+1-n,\ell}\otimes\nu^\ell\to\overline{\mathbb{Q}}$, $(\varphi_1,\varphi_2) := \int_{G/B(\mathbb{Q})\cdot(\mathcal{D}\cdot\mathcal{U})(\mathbb{A}^f)}\varphi_1\cdot\varphi_2\cdot|\det|_f^\ell d\mu$. Now for $\omega\in E$, $\epsilon\in E$ consider the function $\langle\omega,\epsilon\rangle_S : g\mapsto[g^{-1}\omega,g^{-1}\epsilon]_S$; clearly $\langle\omega,\epsilon\rangle_S\in\mathcal{B}_{-s+\ell+2,\ell}$. So we have the G-map $\langle\ \rangle_S : V\otimes_{\overline{\mathbb{Q}}}E\to\mathcal{B}_{-s+\ell+2,\ell}$. For any $\varphi\in\mathcal{B}_{s-1,\ell}$ one has

(4.3.3) $$(\langle\omega,\epsilon\rangle_s,\varphi) = \int_{G/Z(\mathbb{Q})\cdot U(\mathbb{A}^f)} W(\omega)\cdot\overline{W}(\epsilon)\varphi|\det|^\ell d\mu' .$$

Here Z = Center GL_2 and $d\mu'$ is an invariant measure on $G/Z(\mathbb{Q})\cdot\mathcal{U}(\mathbb{A}^f)$; note that $W(\omega)\cdot\overline{W}(\epsilon)\cdot\varphi|\det|^\ell$ is a right $Z(\mathbb{Q})\cdot\mathcal{U}(\mathbb{A}^f)$-invariant function.

Since the spaces $\mathcal{B}_{s,\ell}$ for $s\neq\ell/2,\ \ell/2+1$ are irreducible, 4.3.2 implies that for $n\in\mathbb{Z}$, $n>\frac{\ell+4}{2}$ one has

(4.3.4) $\langle\ \rangle_n(V\otimes E) = L(V,E,n)\cdot L(\theta(V)\cdot\theta(E),2n-\ell-2)^{-1}\cdot\mathcal{B}^0_{-n+\ell+2,\ell}$

or

(4.3.5) the $\overline{\mathbb{Q}}$-space generated by $(\langle\omega,e\rangle_n,\varphi)$, $\omega\in V$, $e\in E$, $\varphi\in\mathcal{B}^0_{n-1,\ell}$ is $L(V,E,n)\cdot L(\theta(V)\cdot\theta(E),2n-\ell-2)^{-1}\cdot\overline{\mathbb{Q}}$.

Remark 4.3.6. The equality 4.3.4 holds also for $n = \frac{\ell+4}{2}$ if we assume that $V\neq E^*$ (for $\ell=0$). (Note that the left-hand side is contained in the right-hand one, and the only non-trivial G-subspace of $\mathcal{B}^0_{\ell/2,\ell}$ is $|\det|^{-\ell/2}\cdot\overline{\mathbb{Q}}$. So, if the equality does not hold we have a non-trivial G-pairing between V and E.) For $n=\frac{\ell+4}{2}$ the space $\mathcal{B}^0_{n-1,\ell}$ contains a unique non-trivial G-subspace $\mathcal{F}^\ell$; clearly 4.3.4 implies that the stronger version of 4.3.5 - when φ runs only through $\mathcal{F}^\ell$-holds. □

Now we may return to Eisenstein series. For $\varphi \in \mathfrak{B}_{s,\ell}$ the section $(2\pi y)^{s-\ell}\cdot\varphi\cdot\mathcal{X}^\ell$ of $\omega^\ell_{C^\infty}(H^+ \times G)$ is $B(\mathbb{Q})^+$-invariant and is is an element of $\omega^\ell_{C^\infty}(\widetilde{M} \otimes \mathbb{C})$. The series $\mathcal{E}^{s,\ell}(\varphi)$ $:= p_*((2\pi y)^{s-\ell}\cdot\varphi\cdot\mathcal{X}^\ell)$ converges absolutely for $\operatorname{Re} s > \frac{\ell+2}{2}$ and $\mathcal{E}^{s,\ell}:\mathfrak{B}_{s,\ell} \to \omega^\ell_{C^\infty}(M)$ commutes with G-action; the map $\mathcal{E}^{\ell+1,\ell}$ is $(\ell+1)\cdot\mathcal{E}^\ell$ from 2.3. So for $e \in E$ we have $C \otimes \overline{\mathbb{Q}}$-valued (1,0)-form $e\cdot\mathcal{E}^{s,\ell}(\varphi)$ on M. Let us compute the scalar product $(\omega, e\cdot\mathcal{E}^{s,\ell}(\varphi))$. We have

$$(\omega, \cdot\mathcal{E}^{s,\ell}(\varphi)) = 2^{\ell+1}\int_{\widetilde{M}(\mathbb{C})}\omega_{(q)}\cdot\overline{}_{(q)}(2\pi y)^s\cdot\varphi(g)\cdot|\det g|^\ell d2\pi y\cdot dx$$

$$\overset{(4.3.1)}{=} 2^{\ell+1}\int_{H^+\times G/Z(\mathbb{Q})\cdot\mathcal{U}(\mathbb{Q})} W(\omega)q\overline{\mathcal{E}}_{(q)}\cdot(2\pi y)^s$$

$$\cdot\varphi(g)|\det g|^\ell d2\pi y\cdot dx .$$

To compute this first integrate along x, i.e., along $\mathcal{U}(\mathbb{A})$-orbits. The Fourier orthogonal relations show that we may replace $\overline{\mathcal{E}}_{(q)}$ by $\overline{W(e)\cdot q}$ in this integral; we get

$$(\omega, e\cdot\mathcal{E}^{s,\ell}(\varphi)) = 2^{\ell+1}\cdot\int_{\mathbb{R}^*_+} \ell^{-4\pi y}(2\pi y)\ d2\pi y$$

$$\cdot\int_{G/Z(\mathbb{Q})\mathcal{U}(\mathbb{A}^f)} \varphi\cdot W(\omega)\cdot\overline{W(e)}|\det|du$$

$$\overset{(4.3.3)}{=} c\cdot 2^{-s}\,\Gamma(s+1)(\langle\omega, e\rangle_{s+1}, \varphi)$$

for certain $c \in \mathbb{Q}^*$. So 4.3.5 implies that for $n > \frac{\ell+4}{2}$ the $\overline{\mathbb{Q}}$-space, generated by $(\omega, e\cdot\mathcal{E}^{n-1,\ell}(\varphi))$, $\omega \in V$, $e \in E$, $\varphi \in \mathfrak{B}^0_{n-1,\ell}$ coincides with $L(V,E,n)\cdot L(\theta(V)\cdot\theta(E), 2n-\ell-2)^{-1}\cdot\overline{\mathbb{Q}}$.

For $E = E_\chi$ and $n = \ell+2$ this statement is exactly 4.2.1, since $L(V,E_\chi,s) = L(V,s)L(V\cdot\chi, s-\ell-1)$ and $\theta(E_\chi) = \chi$. This finishes the proof of 1.3 in case $\ell > 0$.

5. CASE $\ell = 0$.

The proof of Theorem 1.3 in case $\ell = 0$ follows the same pattern as in the case $\ell > 0$. We will discuss here the minor changes needed to treat this case.

5.1. The definition of spaces $\mathfrak{F}^0_{\mathbb{R}} \subset H^0_B(\widetilde{M}^\infty, \mathbb{R})$, $\mathfrak{F}^0 \subset H^0_B(\widetilde{M}^\infty, \mathbb{Q})$ and residue maps $\operatorname{Res}^0_B: H^1_B(M, \mathbb{R}(1)) \to \mathfrak{F}^0_{\mathbb{R}}$, $\operatorname{Res}^0_{\mathcal{M}}: H^1_{\mathcal{M}}(M, \mathbb{Q}(1)) \to \mathfrak{F}^0$ goes without changes. Note that $\mathfrak{F}^0_{\mathbb{R}}$ is the space of $\mathbb{R}$-valued measures on $|M^\infty \otimes \mathbb{R}|$, invariant under the action of sufficiently small open $K \subset G$, and $\mathfrak{F}^0$ is the space of $\mathbb{Q}$-valued ones on $|M^\infty|$. So we have canonical maps $\int: \mathfrak{F}^0_{\mathbb{R}} \to \mathbb{R}$, $\mathfrak{F}^0 \to \mathbb{Q}$; let $\overline{\mathfrak{F}}^0_{\mathbb{R}}$, $\overline{\mathfrak{F}}^0$ be the kernel of $\int$. The exact cohomology sequence shows that the image of Res_B is $\overline{\mathfrak{F}}^0_{\mathbb{R}}$; one also has $\operatorname{Im}\operatorname{Res}_{\mathcal{M}} = \overline{\mathfrak{F}}^0$ by Manin-Drinfeld theorem (note that $H^1_{\mathcal{M}}(M, \mathbb{Q}(1)) = \mathcal{O}^*(M) \otimes \mathbb{Q}$ and $\operatorname{Res}^0_{\mathcal{M}}$ is divisor map).

The forms $\mathcal{X}^0$ and $\mathcal{X}^0_{\mathcal{H}}$ of 2.1.3, 2.2 are $d \log q$ and $\log|q|$ respectively. The Eisenstein series $P_*(\varphi\cdot\mathcal{X}^0_{\mathcal{H}})$ for $\varphi \in \overline{\mathfrak{F}}^0_{\mathbb{R}}$ does not converge absolutely. They are defined as analytic continuation of series $\mathcal{E}^{s,0}$ (see 4.3) to $s = 1$; one proceeds in the same manner with E^0 or put directly $E^0(\varphi) = dz\,\mathcal{E}^0(\varphi)$. The results of 2.1.3, 2.2 and 2.3 remain valid (with $\mathfrak{F}^\ell_{\mathbb{R}}$ replaced by $\overline{\mathfrak{F}}^0_{\mathbb{R}}$).

As in 2.1.3, put $\Phi^0_{\mathbb{R}} = H^1_B(M, \mathbb{R}(1)) \cap F^1H^1_{DR}(M \otimes \mathbb{R})$ $= \operatorname{Im}(H^1_{\mathcal{H}}(M, \mathbb{R}(1)) \to H^1_{\mathbb{R}}(M, \mathbb{R}(1)))$. We have canonical direct sum decomposition (valid for any curve) $H^2_{\mathcal{H}}(M, \mathbb{R}(2)) = H^1_B(M, \mathbb{R}(1))$ $= \Phi^0_{\mathbb{R}} \oplus H^1_B(\overline{M}, \mathbb{R}(1))$. The Lemma 2.4.1 in case $\ell = 0$ is not true as stated but the proof (trivial in this case) shows that $E^0(\operatorname{Res}\psi_1) \cup E^0(\operatorname{Res}\psi_2)$ is the projection of $\psi_1 \cup \psi_2$ to $H^1_B(\overline{M}, \mathbb{R}(1))$. Consider the Poincare duality pairing $\langle\ \rangle: H^1_{B_c}(M, \mathbb{C}) \otimes H^1_B(M, \mathbb{C}) \to \mathbb{R}$ restricted to $\Omega^1(\overline{M} \otimes \mathbb{R}) \subset H^1_{B_c}(M, \mathbb{C}), H^1_B(M, \mathbb{R}(1)) \subset H^1_B(M, \mathbb{C})$. The above shows that $\langle\omega, \psi_1 \cup \psi_2\rangle = \langle\omega, E^0(\operatorname{Res}\psi_1) \cup E^0(\operatorname{Res}\psi_2)\rangle$ $= -(\omega, E^0(\operatorname{Res}\psi_1)\mathcal{E}^0(\operatorname{Res}\psi_2))$.

5.2. The above decomposition of $H^2_{\mathcal{H}}(M, \mathbb{R}(2))$ also holds for $H^2_{\mathcal{M}}$. To see this recall the following lemma of Bloch. Let S be a spectrum of somewhat localized ring of integers in a number field, $p:\overline{C} \to S$ be a projective curve over S and

$C^\infty \subset \overline{C}$ be a divisor. Suppose that $\overline{C}$ is regular scheme and C^∞ is a disjoint union of components S_i s.t. for any i the projection $p|_{C_j}: S_j \to S$ is isomorphism and any $S_i - S_j$ has finite order in $\mathrm{Pic}(\overline{C})$. Put $C := \overline{C} \setminus C^\infty$.

Lemma 5.2.1. Put $\Phi_{\mathcal{M}}(C) := \{H^1_{\mathcal{M}}(C,\mathbb{Q}(1)), p^* H^1_{\mathcal{M}}(S,\mathbb{Q}(1))\} \subset H^2_{\mathcal{M}}(C,\mathbb{Q}(2))$. We have the direct sum decomposition

$$H^2_{\mathcal{M}}(C,\mathbb{Q}(2)) = \Phi(C) \oplus H^2_{\mathcal{M}}(\overline{C},\mathbb{Q}(2)).$$

Proof: Consider the exact localization sequence

$$\oplus H^0_{\mathcal{M}}(S_i,\mathbb{Q}(1)) \to H^2_{\mathcal{M}}(\overline{C},\mathbb{Q}(2)) \xrightarrow{j^*} H^2_{\mathcal{M}}(C,\mathbb{Q}(2))$$
$$\xrightarrow{\partial} \oplus H^1_{\mathcal{M}}(S_i,\mathbb{Q}(1)) \xrightarrow{i_*} H^3_{\mathcal{M}}(\overline{C},\mathbb{Q}(2)) .$$

Since $H^0_{\mathcal{M}}(S_i,\mathbb{Q}(1)) = 0$, the arrow j^* is injective. Since $p_* i_* = \bigoplus_j (p_j|_{C_j})_*$ the image of ∂ is contained in the kernel of the sum of the coordinate map; but this one equals $\partial\Phi(C)$ by our conditions. So $\mathrm{Im}\,\partial = \mathrm{Im}\,\partial|\Phi$, i.e., $H^2_{\mathcal{M}}(C,\mathbb{Q}(2))$ is the sum of our subspaces. To see that this sum is direct, we have to show that $\mathrm{Ker}\,\partial|\Phi(C) = 0$. Note that $\mathrm{Ker}(\partial\circ\{\cdot,\cdot\} = \mathrm{div}\cdot p^*: \mathcal{O}^*(C) \otimes \mathcal{O}^*(S) \otimes \mathbb{Q} \to \bigoplus_i \mathcal{O}^*(S_i) \otimes \mathbb{Q})$ obviously coincides with $\mathcal{O}^*(S) \otimes \mathcal{O}^*(S) \otimes \mathbb{Q}$. Since its image in Φ is zero by Borel's theorems $\{\mathcal{O}^*(S),\mathcal{O}^*(S)\} \otimes \mathbb{Q} \subset K_2(S) \otimes \mathbb{Q} = 0$, the map $\partial|\Phi$ is injective. □

Put $H^2_{\mathcal{M}}(C,\mathbb{Q}(2))^{\mathrm{parab}} = \{\mathcal{O}(C)^*,\mathcal{O}(C)^*\}\cdot\mathbb{Q} \subset H^2_{\mathcal{M}}(C,\mathbb{Q}(2))$, $H^2_{\mathcal{M}}(\overline{C},\mathbb{Q}(2))^{\mathrm{parab}} = H^2_{\mathcal{M}}(\overline{C},\mathbb{Q}(2)) \cap H^2(C,\mathbb{Q}(2))^{\mathrm{parab}}$. Clearly 5.2.1 implies

5.2.2 One has direct sum decomposition

$$H^2_{\mathcal{M}}(C,\mathbb{Q}(2))^{\mathrm{parab}} = \Phi(C) \oplus H^2_{\mathcal{M}}(\overline{C},\mathbb{Q}(2))^{\mathrm{parab}} .$$

Let us apply the above considerations to the fiber $C^\infty_\eta \subset \overline{C}_\eta \supset C_\eta$ of the above situation over the generic point $\eta \in S$. Since $\mathcal{O}^*(C_\eta) = \mathcal{O}^*(\eta)\cdot\mathcal{O}^*(C)$ one has

5.2.3. The restriction map $H^2_{\mathcal{M}}(\overline{C},\mathbb{Q}(2))^{\mathrm{parab}} \to H^2_{\mathcal{M}}(\overline{C}_\eta,\mathbb{Q}(2))^{\mathrm{parab}}$ is surjective.

By Manin-Drinfeld the curves $M_n/\mathbb{Q}[\zeta_n]$ fit into 5.2.1.

5.2.4. One has canonical direct sum decompositions $H^2_{\mathcal{M}}(M,\mathbb{Q}(2)) = H^2_{\mathcal{M}}(\overline{M},\mathbb{Q}(2)) \oplus \Phi_{\mathcal{M}}(M)$, $H^2_{\mathcal{M}}(M,\mathbb{Q}(2))^{\mathrm{parab}} = H^2_{\mathcal{M}}(\overline{M},\mathbb{Q}(2))^{\mathrm{parab}} \oplus \Phi_{\mathcal{M}}(M)$.

Now let us prove that $H^2_{\mathcal{M}}(\overline{M},\mathbb{Q}(2))^{\mathrm{parab}} \subset H^2_{\mathcal{M}}(\overline{M},\mathbb{Q}(2))_{\mathbb{Z}}$ (Theorem 1.2.3; case $\ell = 0$). Consider canonical model of $\overline{M}$ over $\mathbb{Z}$. Namely, let $\overline{M}_{n\mathbb{Z}}$ be the integral closure of $\overline{M}_{0\mathbb{Z}} = \mathbb{P}^1_{\mathbb{Z}}$ in $\overline{M}_n$. Clearly $\overline{M}_{n\mathbb{Z}}$ is a proper scheme over $\mathbb{Z}[\zeta_n]$ with $GL_2(Z/nZ)$-action. For n_1/n_2 one has an obvious map $\overline{M}_{n_2\mathbb{Z}} \to \overline{M}_{n_1\mathbb{Z}}$; put $\overline{M}_{\mathbb{Z}} := \varprojlim \overline{M}_{n\mathbb{Z}}$. One knows that the $\overline{M}_{n\mathbb{Z}}$ are regular schemes, and the scheme $M^\infty_{n\mathbb{Z}} :=$ the closure of M^∞_n in $\overline{M}_{n\mathbb{Z}}$ is a disjoint union of components that project isomorphically to $\mathrm{Spec}\,\mathbb{Z}[\zeta_n]$ (cf. e.g. [5]). So 1.2.3 will follow from 5.2.2, if we prove the following more precise version of Manin-Drinfeld theorem:

Lemma 5.2.4. The difference of any two components of $M^\infty_{n\mathbb{Z}}$ has finite order in $\mathrm{Pic}\,\overline{M}_{n\mathbb{Z}}$.

Proof: Note that for any $x \in \mathrm{Spec}\,\mathbb{Z}[\zeta_n]$ the fiber $\overline{M}_{nx}$ of $\overline{M}_{n\mathbb{Z}}$ over x is reduced. So (by the ordinary Manin-Drinfeld and since $\mathrm{Pic}\,\mathbb{Z}[\zeta_n] \otimes \mathbb{Q} = 0$) 5.2.4 follows from

5.2.4.1. For any $f \in \mathcal{O}^*(M)$ and a closed point $x_{(n)} \in \mathrm{Spec}\,\mathbb{Z}[\zeta_n]$ the order of $\mathrm{div}\,f$ along the irreducible components of $\overline{M}_x$ is constant.

Let $p = \mathrm{char}\,x_{(n)}$ and $n = p^a m$, $(m,n) = 1$. Recall that the components of $\overline{M}_{nx}$ are in natural 1-1 correspondence with points of $\mathbb{P}^1(Z/p^aZ)$; the action of $SL_2(Z/nZ)$ on the first set corresponds to the obvious action via $SL_2(Z/nZ) \to SL_2(Z/p^aZ)$ on the second. It will be convenient for us to pass to $\overline{M}$: so, if $x \in \mathrm{Spec}\,\mathbb{Q}[\zeta]$ is closed point of char p then, the set of components of M_x is $\mathbb{P}^1(\mathbb{Z}_p) = \mathbb{P}^1(\mathbb{Q}_p)$. The natural action of G on $\overline{M}$ prolongs to the one on $\overline{M}_{\mathbb{Z}}$; the group $SL_2(A^f) \subset G$ acts on the set of components of $\overline{M}_x$ via the projection

$$SL_2(A^f) \to SL_2(\mathbb{Q}_p) \to \mathrm{Aut}\,\mathbb{P}^1(\mathbb{Q}_p).$$

These facts easily imply 5.2.4.1. Namely, for a profinite set L denote by $\mathfrak{F}(L)$ the space of locally-constant $\mathbb{Q}$-valued functions on L, let $\mathbb{Q} \subset L$ be the space of constant functions. The arrows $\mathrm{div}_{x_n}: \mathcal{O}^*(M_n) \to \mathfrak{F}(\mathbb{P}^1(\mathbb{Z}/p^a\mathbb{Z}))$, $\mathrm{div}_{x_n}(f)(\alpha) = \frac{\mathrm{ord}_\alpha f}{\mathrm{ord}_{x_n} p}$ are compatible for different n and so defined $SL_2(\mathbb{A}^f)$-map $\mathrm{div}_x: \mathcal{O}^*(M) \otimes \mathbb{Q} \to \mathfrak{F}(\mathbb{P}^1(\mathbb{Q}_p))$. We have to show that $\mathrm{div}_x(\mathcal{O}^*(M) \otimes \mathbb{Q}) \subset \mathbb{Q}$. But the $SL_2(\mathbb{A}^f)$-module $\mathcal{O}^*(M) \otimes \mathbb{Q}$ is an automorphic representation (Eisenstein series + trivial module). The space $\mathfrak{F}(\mathbb{P}^1(\mathbb{Q}_p))/\mathbb{Q}$ is the sum of infinite-dimensional representations of $SL_2(\mathbb{Q}_p)$, and $SL_2(\mathbb{A}^f)$ acts on it via the projection on $SL_2(\mathbb{Q}_p)$. So $\mathrm{Hom}_{SL_2(\mathbb{A}^f)}(\mathcal{O}^*(M) \otimes \mathbb{Q}, \mathfrak{F}(\mathbb{P}^1(\mathbb{Q}_p))/\mathbb{Q}) = 0$, and we are done. □

5.3. All the results of Section 4 remain valid for $\ell = 0$: one has to use Remark 4.3.6. This finishes the proof in case $\ell = 0$.

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NOTES ON ABSOLUTE HODGE COHOMOLOGY

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INTRODUCTION. First a few words about the situation in étale cohomology to motivate what follows. Let $\pi: X \to \mathrm{Spec}\ K$ be a scheme over a field K, and let $\mathfrak{F} \in D^b(X_{\acute{e}t})$ be a complex of sheaves on $X_{\acute{e}t}$; put $\underline{R\Gamma}(X,\mathfrak{F}) := R\pi_*(\mathfrak{F}) \in D^b((\mathrm{Spec}\ K)_{\acute{e}t})$. We have

$$R\Gamma(X_{\acute{e}t},\mathfrak{F}) = R\Gamma((\mathrm{Spec}\ K)_{\acute{e}t}, \underline{R\Gamma}(X,\mathfrak{F})) = R\ \mathrm{Hom}_{D^b((\mathrm{Spec}\ K)_{\acute{e}t})}(\mathbb{Z}, \underline{R\Gamma}(X,\mathfrak{F})).$$

If $\bar{K}/K$ is a separable closure of K and $G = \mathrm{Gal}\ \bar{K}/K$, then the sheaves on $(\mathrm{Spec}\ K)_{\acute{e}t}$ are G-modules, and $\underline{R\Gamma}(X,\mathfrak{F}) = R\Gamma((X \otimes_K \bar{K})_{\acute{e}t}, \mathfrak{F})$ is the geometric étale cochain complex of X with canonical G-action.

0.1. Now suppose that $K = \mathbb{C}$. Then, following Deligne [4] the role of sheaves on "arithmetic" Spec $\mathbb{C}$ should be played by (mixed) Hodge structures. This analogy suggests that for any scheme X there should be a canonical object $\underline{R\Gamma}(X,\mathbb{Z}) \in D^b(\mathcal{H})$ (where $\mathcal{H}$ = category of Hodge structures), whose underlying complex of abelian groups is the usual chain complex of topological space $X(\mathbb{C})$. We will see that this is indeed the case: The basic construction of Deligne [4] plus a bit of homological algebra do the job. For $i \in \mathbb{Z}$ define the absolute Hodge cochain complex of X with coefficients in $\mathbb{Z}(i)$

$$R\Gamma_{\mathcal{H}}(X,\mathbb{Z}(i)) := R\ \mathrm{Hom}_{D^b(\mathcal{H})}(\mathbb{Z}, \underline{R\Gamma}(X,\mathbb{Z})(i)).$$

Here (i) on the right-hand side means Tate twist in $D^b(\mathcal{H})$. The absolute Hodge (or simply $\mathcal{H}$-) cohomology groups $H^{\cdot}_{\mathcal{H}}(X,\mathbb{Z}(*)) = H^{\cdot}(R\Gamma_{\mathcal{H}}(X,\mathbb{Z}(*)))$ form a twisted Poincaré duality theory in the sense of [3]. They may be easily computed in terms of Deligne-Hodge structure in $H^{\cdot}(X)$, e.g. we have canonical