EISENSTEIN CLASS OF A TORUS BUNDLE AND LOG-RIGID ANALYTIC CLASSES FOR $\mathrm{SL}_n(\mathbb{Z})$

MARTI ROSET AND PETER XU

Abstract. Departing from a topological treatment of the Eisenstein class of a torus bundle, we define log-rigid analytic classes for $\mathrm{SL}_n(\mathbb{Z})$. These are group cohomology classes for $\mathrm{SL}_n(\mathbb{Z})$ valued on log-rigid analytic functions on Drinfeld's p-adic symmetric domain. Such classes can be evaluated at points attached to totally real fields of degree n where p is inert. We conjecture that these values are p-adic logarithms of Gross–Stark units in the narrow Hilbert class field of totally real fields. We provide evidence for the conjecture by comparing our constructions to p-adic L-functions. In addition, we prove it in certain situations where the totally real field is Galois over $\mathbb{Q}$, as a consequence of the fact that in this case there is a conjugate of a Gross–Stark unit in $\mathbb{Q}_p$.

Contents

1. Introduction

The values of modular units at CM points, called elliptic units, have rich arithmetic significance. Notably, they generate abelian extensions of imaginary quadratic fields. In [DD], Darmon and Dasgupta proposed a conjectural construction of elliptic units for real quadratic fields and predicted that they behave similarly to elliptic units. Their construction consists of a p-adic limiting process involving periods of logarithmic derivatives of modular units along real quadratic geodesics.

Using a different approach, Dasgupta used Shintani's method to extend this construction to totally real fields of arbitrary degree in [Das08], and Dasgupta and Kakde proved that this recipe gives p-units in abelian extensions of totally real fields [DK23], [DK24]. More precisely, they proved that their resulting objects are Gross–Stark units. Remarkably, their work provides a solution to Hilbert's twelfth problem for totally real fields via p-adic methods.

Darmon, Pozzi, and Vonk constructed analogs of modular functions, called rigid classes, which can be evaluated at real quadratic points, and expressed the original construction of [DD] as the value of a rigid class in [DPV24]. Their work accentuates the parallel between Gross–Stark units and elliptic units, as rigid classes play the role of modular functions. Moreover, it leads to an alternative proof of the conjecture of [DD] in the real quadratic setting.

In this paper, we construct a log-rigid analytic class for $\mathrm{SL}_n(\mathbb{Z})$ and study its values at points attached to totally real fields where p is inert. We conjecture that these values are p-adic logarithms of Gross–Stark units and provide evidence for it by comparing our constructions to p-adic L-functions and using the rank 1 Gross–Stark conjecture, proven in [DDP11] and [Ven15].

Moreover, we prove the conjecture in certain situations where the totally real field is Galois over $\mathbb{Q}$, as a consequence of the fact that in this case there is a conjugate of a Gross–Stark unit in $\mathbb{Q}_p$. To our knowledge, this extends the type of abelian extensions of F that can be constructed explicitly using only the values of the derivatives of p-adic L-functions. Moreover, it expresses Gross–Stark units as values of a modular-like object, namely the log-rigid class.

A key ingredient in our construction is the Eisenstein class of a torus bundle of Bergeron, Charollois, and García [BCG20] and its pullbacks by torsion sections, that replace the role of modular units. In particular, we conjecture that Gross–Stark units can be obtained via a p-adic limiting process involving periods of Eisenstein classes on locally symmetric spaces attached $\mathrm{SL}_n(\mathbb{R})$ along tori determined by totally real fields. We hope that this represents a first step toward a modular, or more technically automorphic, construction of Gross–Stark units for totally real fields, which would generalize the results of [DD] and [DPV24].

1.1. **Siegel units and abelian extensions of quadratic fields.** We begin by explaining the construction of Siegel units and their relation with the theory of complex multiplication for imaginary quadratic fields. Let E be an elliptic curve defined over a scheme S, fix a positive integer c coprime to 6, and denote by $\mathbb{N}^{(c)}$ the set of positive integers coprime to c.

Proposition 1.1. *There exists a unique function ${}_c\theta_E \in \mathcal{O}(E - E[c])^\times$ satisfying:*

(1) The divisor of ${}_c\theta_E$ is $E[c] - c^2(0)$.
(2) ${}_c\theta_E$ is invariant under pushforward induced by multiplication by a for all $a \in \mathbb{N}^{(c)}$.

Let $N \geq 3$ be a positive integer coprime to c, denote by $\Gamma(N) \subset \mathrm{SL}_2(\mathbb{Z})$ the congruence subgroup of full level N, and let $\mathscr{H}$ be the complex upper half-plane. We can then consider the universal elliptic curve

$$E := \Gamma(N)\backslash\left((\mathscr{H} \times \mathbb{C})/\mathbb{Z}^2\right) \longrightarrow Y(N) := \Gamma(N)\backslash\mathscr{H}.$$

The proposition above yields the function ${}_c\theta_E \in \mathcal{O}(E-E[c])^\times$, which can be used to construct modular units on $\Gamma(N)\backslash\mathscr{H}$ in the following way. A vector $v \in \mathbb{Q}^2/\mathbb{Z}^2 - \{0\}$ of order N induces a torsion section $v\colon Y(N) \to E - E[c]$. Then, the pullback ${}_cg_v := v^*({}_c\theta_E) \in \mathcal{O}(Y(N))^\times$ is called a *Siegel unit* and is an instance of a modular unit. It gives rise to a $\Gamma(N)$-invariant function on $\mathscr{H}$, that we will denote by the same symbol. The theory of complex multiplication implies that the values of Siegel units at special points have deep significance.

Theorem 1.2. *Let $\tau \in \mathscr{H}$ be a CM point attached to a quadratic imaginary field K, i.e. τ is stabilized by a subgroup of norm one elements $K^1 \subset \mathrm{SL}_2(\mathbb{Q})$ of K. Then,*

$${}_c g_v(\tau) \in K^{\mathrm{ab}} \subset \bar{\mathbb{Q}}.$$

An important question is to find an analog of this theorem for general number fields. The case of real quadratic fields has been extensively studied via different methods. We are particularly interested in the p-adic approach initiated by Darmon and Dasgupta in [DD] and followed, among others, by Darmon, Pozzi, and Vonk in [DPV24]. We proceed to outline these works in a language suited to this paper.

Let F be a real quadratic field and p a rational prime. Observe that $\mathscr{H}$ does not contain real quadratic points, i.e. there are no points stabilized by a torus of norm one elements $F^1 \subset \mathrm{SL}_2(\mathbb{Q})$ of F. On the other hand, $\mathscr{H}$ has geodesics stabilized by these tori. Moreover, if $(z, \gamma z) \subset \mathscr{H}$ is a segment of such geodesic, where $\gamma \in F^1 \cap \Gamma(p^r)$, and $v \in \mathbb{Q}^2/\mathbb{Z}^2 - \{0\}$ is of exact order p^r, we have the so-called Meyer's theorem

$$\frac{1}{2\pi i}\int_z^{\gamma z} \mathrm{dlog}({}_c g_v) = \zeta_c(F, [\mathfrak{b}], 0) \in \mathbb{Z}. \tag{1}$$

Here, $\zeta_c(F, [\mathfrak{b}], 0)$ denotes the value at $s = 0$ of a c-smoothed partial zeta function attached to F and an ideal class $[\mathfrak{b}]$ in a narrow class group of conductor divisible p^r, determined by the inclusion $F^1 \subset \mathrm{SL}_2(\mathbb{Q})$ and v. In addition to encoding information about abelian extensions of totally real fields, these zeta values possess notable p-adic properties and serve for the construction of measures that yield p-adic partial zeta functions of F.

The search for a symmetric space containing real quadratic points, combined with the p-adic properties of the partial zeta values considered above, leads to replacing $\mathscr{H}$ by a p-adic symmetric space to generalize Theorem 1.2. More precisely, if we let $\mathscr{H}_p := \mathbb{P}^1(\mathbb{C}_p) - \mathbb{P}^1(\mathbb{Q}_p)$ be the p-adic upper half-plane and $\mathcal{A}$ its ring of rigid analytic functions, we have:

- $\mathscr{H}_p$ contains points stabilized by $F^1 \subset \mathrm{SL}_2(\mathbb{Q})$ if and only if p is nonsplit in F.
- There is a $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant isomorphism between $\mathcal{A}^\times/\mathbb{C}_p^\times$ and the space of $\mathbb{Z}$-valued measures on $\mathbb{P}^1(\mathbb{Q}_p)$ of total mass zero ([Put82]), suggesting that $\mathcal{A}^\times$ encodes information about p-adic zeta functions and refinements of their values.

In [DPV24], Darmon, Pozzi, and Vonk exploit the distribution relation of Siegel units attached to vectors of arbitrary p-power order to construct a cohomology class

$$\mathcal{J}_{\mathrm{DR}} \in H^1(\mathrm{SL}_2(\mathbb{Z}), \mathcal{A}^\times).$$

This class can be viewed as a generalization of a modular function. Indeed, the space of invariant functions $H^0(\mathrm{SL}_2(\mathbb{Z}), \mathcal{A}^\times) = \mathbb{C}_p^\times$ is too simple, which suggests studying the first cohomology group instead. Moreover, if $\tau \in \mathscr{H}_p$ is stabilized by $F^1 \subset \mathrm{SL}_2(\mathbb{Q})$ and $F^1 \cap \mathrm{SL}_2(\mathbb{Z}) = \langle \pm\gamma_\tau \rangle$, they define the value $\mathcal{J}_{\mathrm{DR}}[\tau] := \mathcal{J}_{\mathrm{DR}}(\gamma_\tau)(\tau) \in \mathbb{C}_p^\times$.

Theorem 1.3 (Darmon–Pozzi–Vonk). *Let $\tau \in \mathscr{H}_p$ be as above with stabilizer $\langle \pm\gamma_\tau \rangle \subset \mathrm{SL}_2(\mathbb{Z})$ be attached to a real quadratic field F where p is inert, and suppose that $\{\tau, 1\}$ generate a fractional ideal of F. Then,*

$$\log_p(\mathcal{J}_{\mathrm{DR}}[\tau]) = \log_p(u), \; u \in H = \textit{narrow Hilbert class field of } F \subset F^{\mathrm{ab}} \subset \bar{\mathbb{Q}}.$$

This theorem provides a level 1 version of Theorem 1.2 for real quadratic fields where p is inert. Indeed, it produces nontrivial elements in abelian extensions of real quadratic fields as values of $\mathscr{J}_{\mathrm{DR}}$ at special points in $\mathscr{H}_p$.

Remark 1.4. The class $\mathscr{J}_{\mathrm{DR}}$ is the unique lift via the quotient map $\mathcal{A}^\times \to \mathcal{A}^\times/\mathbb{C}_p^\times$ of the restriction to $\mathrm{SL}_2(\mathbb{Z})$ of a class $J_{\mathrm{DR}} \in H^1(\mathrm{SL}_2(\mathbb{Z}[1/p]), \mathcal{A}^\times/\mathbb{C}_p^\times)$, also constructed in [DPV24]. Moreover, the Hecke module $H^1(\mathrm{SL}_2(\mathbb{Z}[1/p]), \mathcal{A}^\times/\mathbb{C}_p^\times)_\mathbb{Q}$ is isomorphic to the sum of $H^1(\Gamma_0(p), \mathbb{Q})$ and an Eisenstein line. The lift $\mathscr{J}_{\mathrm{DR}}$ is important to define the values of J_{DR} and it is the object we aim to generalize in this work. Here and for the rest of the paper, the subindex $\mathbb{Q}$ denotes the tensor product with $\mathbb{Q}$ over $\mathbb{Z}$.

1.2. **Construction of the log-rigid class for** $\mathrm{SL}_n(\mathbb{Z})$**.** The work of Bergeron, Charollois, and García in [BCG20] provides a generalization of logarithmic derivatives of Siegel units which is relevant for the study of totally real fields of degree n: the *Eisenstein class of a torus bundle*. Let $E \to X$ be an oriented real vector bundle of rank n over an oriented manifold X. Suppose that E contains a sub-bundle $E_\mathbb{Z}$ with fibers isomorphic to $\mathbb{Z}^n$. We can then construct the torus bundle $T := E/E_\mathbb{Z} \to X$. Consider the following class in singular cohomology with $\mathbb{Z}$-coefficients

$$T[c] - c^n\{0\} \in H^0(T[c]) \simeq H^n(T, T - T[c]),$$

where the isomorphism above is the Thom isomorphism. The long exact sequence in relative cohomology provides a map $H^{n-1}(T - T[c]) \to H^n(T, T - T[c])$. The Eisenstein class ${}_cz_T$ attached to T and c is constructed from the next theorem and is analogous to the functions ${}_c\theta_E$ determined in Proposition 1.1.

Theorem 1.5 ([BCG20])**.** *There exists a unique class* ${}_cz_T \in H^{n-1}(T - T[c], \mathbb{Z}[1/c])$ *satisfying:*

(1) ${}_cz_T$ *is a lift of* $T[c] - c^n\{0\} \in H^n(T, T - T[c], \mathbb{Z}[1/c])$.

(2) ${}_cz_T$ *is invariant under pushforward induced by multiplication by a for all* $a \in \mathbb{N}^{(c)}$.

Let $\mathscr{X} := \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n$ be the symmetric space attached to $\mathrm{SL}_n(\mathbb{R})$, let $v_r \in \mathbb{Q}^n/\mathbb{Z}^n$ be the column vector $(1/p^r, 0, \ldots, 0)^t$ and let Γ_r be its stabilizer in $\Gamma := \mathrm{SL}_n(\mathbb{Z})$. Finally, fix q an auxiliary integer such that the full level congruence subgroup $\Gamma(q)$ is torsion-free and $[\Gamma : \Gamma(q)]$ is prime to p, which imposes that p is sufficiently large. Then, $\Gamma_r(q) := \Gamma_r \cap \Gamma(q)$ is torsion-free. We can apply the previous theorem to the universal family of tori

$$T_r := \Gamma_r(q)\backslash(\mathscr{X} \times \mathbb{R}^n/\mathbb{Z}^n) \longrightarrow \Gamma_r(q)\backslash\mathscr{X}$$

and obtain the Eisenstein class ${}_cz_{T_r}$, that we will simply denote by z_r. The vector v_r induces a torsion section $v_r\colon \Gamma_r(q)\backslash\mathscr{X} \to T_r - T_r[c]$ and we can consider the pullback $v_r^*z_r$, which defines a Γ_r-invariant class on $\Gamma_r(q)\backslash\mathscr{X}$. This class is a higher-dimensional analog of $\mathrm{dlog}_c g_v$.

The pullbacks of Eisenstein classes by p-power torsion sections satisfy distribution relations parallel to those of Siegel units. In particular, $(v_r^*z_r)_r$ are compatible with respect to pushforward by the projection maps. Using these properties and Shapiro's lemma, we package the pullbacks of the Eisenstein classes by p-power torsion sections in a group cohomology class

$$\mu_0 \in H^{n-1}(\Gamma, \mathbb{D}_0(\mathbb{X}, \mathbb{Z}[1/m]))^{w=-1},$$

where m is a multiple of c prime to p, $\mathbb{D}_0(\mathbb{X}, \mathbb{Z}[1/m])$ is the space of $\mathbb{Z}[1/m]$-valued measures on $\mathbb{X} := \mathbb{Z}_p^n - p\mathbb{Z}_p^n$ of total mass zero, and w denotes the involution given by the action of $\mathrm{GL}_n(\mathbb{Z})/\mathrm{SL}_n(\mathbb{Z})$. We will sometimes refer to μ_0 as an *Eisenstein cocycle*, following precedent in the literature, which we briefly review and compare with our approach in Section 1.4.

The class μ_0 valued in $\mathbb{D}_0(\mathbb{X}, \mathbb{Z}[1/m])$ is suitable for the construction of rigid classes via a Poisson kernel. Let $\mathscr{X}_p := \mathbb{P}^{n-1}(\mathbb{C}_p) - \bigcup_{H\in\mathcal{H}} H$ be Drinfeld's p-adic symmetric domain, where $\mathcal{H}$ is the set of all $\mathbb{Q}_p$-rational hyperplanes. Denote by $\mathcal{A}_{\mathcal{L}}$ the space of log-rigid analytic functions. Informally, $\mathcal{A}_{\mathcal{L}}$ consists of the $\mathbb{C}_p$-valued functions on $\mathscr{X}_p$ such that its restriction to any affinoid is of the form

$$\text{(rigid analytic function)} + \sum_{H,H'\in H} c_{H,H'} \log_p\left(\ell_H(z)/\ell_{H'}(z)\right),$$

where $c_{H,H'} \in \mathbb{Q}_p$ are all but finitely many 0, $\ell_H(z)$ denotes the equation of the hyperplane $H \in \mathcal{H}$, and $\log_p\colon \mathbb{C}_p^\times \to \mathbb{C}_p$ is the p-adic logarithm satisfying $\log_p(p) = 0$. Integration over $\mathbb{X}$ leads to a Γ-equivariant lift

$$\mathrm{ST}\colon \mathbb{D}_0(\mathbb{X}, \mathbb{Z}_p) \longrightarrow \mathcal{A}_{\mathcal{L}},\ \lambda \longmapsto \left(z \longmapsto \int_{\mathbb{X}} \log_p(z^t \cdot x) d\lambda\right) \tag{2}$$

and we define our desired log-rigid analytic class as

$$J_{E,\mathcal{L}} := \mathrm{ST}(\mu_0) \in H^{n-1}(\Gamma, \mathcal{A}_{\mathcal{L}}).$$

The construction of $J_{E,\mathcal{L}}$ can be compared to that in [DPV24] when $n = 2$, leading to the relation $J_{E,\mathcal{L}} = \log_p(\mathcal{J}_{\mathrm{DR}})$. In particular, this shows that the class $\log_p(\mathcal{J}_{\mathrm{DR}})$ can be constructed solely from logarithmic derivatives of Siegel units, rather than from the full Siegel units.

1.3. **Values of $J_{E,\mathcal{L}}$ at totally real fields where p is inert.** Let F be a totally real field of degree n where p is inert, and let $\tau \in F^n$ be such that its coordinates give an oriented $\mathbb{Z}$-basis $\mathfrak{a}^{-1}$, for $\mathfrak{a}$ an ideal of $\mathcal{O}_F$. Since p is inert, it follows that $\tau \in \mathscr{X}_p$. Moreover, τ is a special point in $\mathscr{X}_p$ in the sense that its stabilizer in $\mathrm{SL}_n(\mathbb{Q})$ is isomorphic to the norm 1 elements of F. In particular, its stabilizer in Γ is a group of rank $n-1$. Following a similar recipe than the case $n = 2$, we define the evaluation of $J \in H^{n-1}(\Gamma, \mathcal{A}_{\mathcal{L}})$ at $\tau \in \mathscr{X}_p$, giving $J[\tau] \in \mathbb{C}_p$. From our construction, one readily deduces $J_{E,\mathcal{L}}[\tau] \in F_p$ and the theorem below gives evidence regarding the arithmetic significance of this value.

Let H be the narrow Hilbert class field of F. Fix an embedding $\bar{\mathbb{Q}} \subset \bar{\mathbb{Q}}_p$, which determines a prime $\mathfrak{p}$ of H above $p\mathcal{O}_F$. Denote by $\mathcal{O}_H[1/p]_-^\times$ the subgroup of p-units of H where every complex conjugation of H acts by -1. Attached to $\mathfrak{p}$ and c, there is a Gross–Stark unit $u \in \mathcal{O}_H[1/p]_-^\times \otimes \mathbb{Q}$, whose valuations at primes above p are related to c-smoothed partial L-functions of the extension H/F. In fact, the proof of the Brumer–Stark conjecture in [DK23] and [Das+23] ensures that $u \in \mathcal{O}_H[1/p]_-^\times$ under certain minor assumptions on c, that we will assume for the rest of the introduction (see Remark 6.3).

Theorem 1.6. *For $n \geq 2$, $\mathrm{Tr}_{F_p/\mathbb{Q}_p} J_{E,\mathcal{L}}[\tau] = \mathrm{Tr}_{F_p/\mathbb{Q}_p} \log_p(u^{\sigma_{\mathfrak{a}}})$, where $u \in \mathcal{O}_H[1/p]_-^\times$ is the Gross–Stark unit given above and $\sigma_{\mathfrak{a}} \in \mathrm{Gal}\,(H/F)$ is the Frobenius corresponding to $\mathfrak{a}$.*

The proof of this result uses that the integral of $v_r^* z_r$ along the $(n-1)$-dimensional submanifold of $\Gamma_r(q)\backslash\mathscr{X}$ determined by the inclusion $F^1 \subset \mathrm{SL}_n(\mathbb{Q})$ is a special value of a partial zeta function of F, generalizing (1). From there, we construct the p-adic partial zeta function of F attached to $\mathfrak{a}$ from μ_0 and express $\mathrm{Tr}_{F_p/\mathbb{Q}_p} J_{E,\mathcal{L}}[\tau]$ as its derivative at $s=0$. Thus, Theorem 1.6 follows from the Gross–Stark conjecture in rank 1, proved in [DDP11] and [Ven15].

The previous theorem, together with Theorem 1.3 involving real quadratic fields, suggests the following conjecture.

Conjecture 1.7. *We have $J_{E,\mathcal{L}}[\tau] = \log_p(u^{\sigma_{\mathfrak{a}}})$, where $u \in \mathcal{O}_H[1/p]^\times_-$ and $\sigma_{\mathfrak{a}} \in \mathrm{Gal}\,(H/F)$ are as above.*

We now outline some evidence towards the conjecture. When $F/\mathbb{Q}$ is Galois, the Gross–Stark units satisfy $\log_p(\sigma_{\mathfrak{a}} u) \in \mathbb{Q}_p$ if the narrow ideal class $[\mathfrak{a}]$ is $\mathrm{Gal}\,(F/\mathbb{Q})$-stable. On the other hand, it can be deduced from the properties of $J_{E,\mathcal{L}}$ that $J_{E,\mathcal{L}}[\tau] \in \mathbb{Q}_p$ if τ generates an ideal $\mathfrak{a}^{-1}$ that is $\mathrm{Gal}\,(F/\mathbb{Q})$-stable. Thus, it follows from Theorem 1.6 that the values of $J_{E,\mathcal{L}}$ can be used to calculate the p-adic logarithm of the Gross–Stark unit u in this setting.

Theorem 1.8. *Suppose $F/\mathbb{Q}$ is Galois, p is inert in F, and τ generates an ideal $\mathfrak{a}$ that is $\mathrm{Gal}\,(F/\mathbb{Q})$-stable. Then, $J_{E,\mathcal{L}}[\tau] = \log_p(u^{\sigma_{\mathfrak{a}}}) \in \mathbb{Q}_p$ for the Gross–Stark unit $u \in \mathcal{O}_H[1/p]^\times_-$ introduced above.*

In particular, Theorem 1.8 applies to real quadratic fields, recovering instances of [DPV24, Theorem B] when the coordinates of τ generate a $\mathrm{Gal}\,(F/\mathbb{Q})$-fixed ideal of $\mathcal{O}_F$. Note that this result implies that we can obtain a formula for certain Gross–Stark units in the narrow Hilbert class field of F only from derivatives of p-adic L-functions in settings where F_p is not equal to $\mathbb{Q}_p$, see Remark 7.4. Moreover, the relevant unit involved in the constructions appears as values of the modular-like object $J_{E,\mathcal{L}}$, supporting the parallel between $J_{E,\mathcal{L}}$ and Siegel units. This extends the type of abelian extensions of F that can be constructed only from derivatives of p-adic L-functions (and note that the field generated by one Gross–Stark unit is Galois over F and therefore contains all its conjugates). This Gross–Stark unit is one of the elements used by Dasgupta–Kakde to construct the maximal abelian extension of F. In ongoing work, we are exploring which ramified abelian extensions of F can be constructed using this observation.

The proof of Conjecture 1.7 would give a construction of Gross–Stark units using Eisenstein classes defined purely from the topology of torus bundles. Moreover, it is possible to find explicit representatives of the classes considered in this article via an integral symbol complex (similarly to the article [Xu25] of the second-named author), which we will present in a sequel to this article. Ultimately, such formulas can be related to those obtained via Shintani's method, as we mention below, and from there to the formulas for Gross–Stark units of [Das08] proven in [DK24]. We hope that this paves the way for proving Conjecture 1.7.

The symbol complex methods mentioned above also seem to shed light on the construction of rigid analytic classes for $\mathrm{SL}_n(\mathbb{Z}[1/p])$ lifting our log-rigid class, which we will present in the sequel. This would fully generalize the construction of J_{DR} of [DPV24] to SL_n, and suggest a different approach to Conjecture 1.7, namely to generalize the strategy of [DPV24].

In a different direction, Darmon, Gehrmann, and Lipnowski generalized the theory of rigid *meromorphic* classes of [DV21] to the setting of orthogonal groups in [DGL25], see [GGM25] for an extensive list of values of these classes at special points.

Remark 1.9. In this paper, we focused on invariants that conjecturally belong to the narrow Hilbert class field of F. It would be interesting to explore the following extensions of the construction. First, we could evaluate $J_{E,\mathcal{L}}$ at points $\tau \in \mathscr{X}_p$ represented by $\tau \in F^n$ whose coordinates give a $\mathbb{Q}$-basis of F, but they do not necessarily span an ideal of F over $\mathbb{Z}$. In this case, we expect to construct invariants in ramified abelian extensions of F. For example, when $n = 2$, such invariants are conjectured to belong to ring class fields of F. Second, when n is odd, Gross–Stark units in the narrow Hilbert class field of F are trivial. It seems that to construct meaningful invariants also in the setting when n is odd, we would need a higher-level version of $J_{E,\mathcal{L}}$, generalizing [Cha09] to totally real fields. For such construction, we expect that the corresponding invariants belong to ray class fields of F.

1.4. **Related cocycles in the literature.** There are constructions of similar cohomology classes to μ_0 in the literature, frequently under the name of *Eisenstein cocycles*. Notably, the work of Sczech [Scz93] together with its integral refinement by Charollois and Dasgupta [CD14, Theorem 4], and the works of Charollois, Dasgupta, Greenberg, and Spiess (see [CDG15] and [DS18]) using Shintani's method give explicit formulas for Eisenstein cocycles. These works yield cocycles for S-arithmetic groups; on the other hand, they take values in measures on $\mathbb{X}$ together with some additional data, such as a set of linear forms in n-variables (used for Q-summation), or the set of rays in $\mathbb{R}^n$ not generated by a vector in $\mathbb{Q}^n$.

More directly related to our approach is the work of Beilinson, Kings, and Levin [BKL18] in the equivariant cohomology of a torus, its adelic refinement by Galanakis and Spiess [GS24], as well as the results of Bannai et. al. in equivariant Deligne cohomology [Ban+24]. These articles also define equivariant Eisenstein classes by specifying residues in a torus bundle but work with larger and more general coefficient modules, such as the logarithm sheaf or a variant of it. In this way, the first two articles construct distribution-valued cohomology classes by delicate topological considerations. Our Eisenstein class is closely related to the specialization to trivial coefficients of these classes (see Remark 4.8). Some computations in cohomology will afford us a lift of our class with finer properties, e.g. a total-mass zero condition, making it sufficient to construct log-rigid classes and produce a conjectural formula for Gross–Stark units.

The latter article [Ban+24] works equivariantly under a nonsplit torus associated to a particular totally real field (rather than a general linear group), and relates a de Rham regulator of this class to L-values closely tied to a method of Shintani. This suggests that the class μ_0, or its restriction to a nonsplit torus, can be compared to the cocycles of [CDG15], [DS18], and [Spi14].

1.5. **Structure of the paper.** In Section 2, we define the Eisenstein class of a torus bundle and prove a distribution relation involving the pullbacks of this class by torsion sections. In Section 3, we introduce an explicit differential form representing the Eisenstein class for a universal family of tori following [BCG20]. We use it to prove that the sum of the pullbacks of this form along the torsion sections of exact order p is 0. The content of these two sections is combined in Section 4 to construct the class $\mu_0 \in H^{n-1}(\Gamma, \mathbb{D}_0(\mathbb{X}, \mathbb{Z}[1/m]))^-$. In Section 5, we construct the log-rigid class $J_{E,\mathcal{L}}$ from μ_0 and define its values at points attached to totally real fields where p is inert. In Section 6, we prove Theorem 1.6, relating the local trace of these values, to the derivative of a p-adic L-function, and therefore to local traces of p-adic logarithms of Gross–Stark units. Finally, in Section 7, we state Conjecture 1.7, and study the case where $F/\mathbb{Q}$ is Galois.

1.6. **Acknowledgements.** We would like to thank Henri Darmon for suggesting that we work on this project. We are also grateful to both him and Pierre Charollois for their constant advice during its development. We are further very grateful to Nicolas Bergeron, Romain Branchereau, Luis García, Lennart Gehrmann, Eyal Goren, Hazem Hassan, Peter Patzt, Brent Pym, and Jan Vonk for stimulating discussions and key suggestions on the subject. We would also like to thank Mateo Crabit Nicolau for his advice when performing numerical computations with Gross–Stark units, and Håvard Damm-Johnsen, Max Fleischer, and Yijia Liu for publicly sharing their algorithms on computing Gross–Stark units for real quadratic fields.

Marti Roset received support from an FRQNT Doctoral Training Scholarship.

2. Eisenstein class of a torus bundle

In this section, we introduce the Eisenstein class of a torus bundle, as studied in [BCG20]. We focus specifically on the torus bundle

$$\Gamma' \backslash (\mathscr{X} \times \mathbb{R}^n/\mathbb{Z}^n) \longrightarrow \Gamma' \backslash \mathscr{X},$$

where $\mathscr{X}$ is the symmetric space attached to $\mathrm{SL}_n(\mathbb{R})$ and $\Gamma' \subset \Gamma := \mathrm{SL}_n(\mathbb{Z})$ is a congruence subgroup that is torsion-free. We then prove several properties of this class, including a distribution relation between its pullbacks by torsion sections, which parallels the distribution relations satisfied by Siegel units. Unless stated otherwise, in this section, we consider singular cohomology with $\mathbb{Z}$-coefficients.

2.1. **Thom and Eisenstein classes of a torus bundle.** Let $\pi\colon E \to X$ be an oriented real vector bundle of rank n over an oriented manifold X. Since E is oriented, for every fiber $E_x \subset E$ over $x \in X$ we have a preferred generator

$$u_{E_x} \in H^n(E_x, E_x - \{0\}) \simeq \mathbb{Z}$$

satisfying a local compatibility condition (see [MS74, Page 96]). The Thom isomorphism theorem asserts that there is a global class that restricts to the orientation of each fiber.

Theorem 2.1 (Thom isomorphism theorem)**.** *There is a unique class $u_E \in H^n(E, E - \{0\})$ such that its pullback to any fiber E_x of E is equal to u_{E_x}. Moreover, for every $i \in \mathbb{Z}$, we have an isomorphism*

$$H^i(X) \xrightarrow{\sim} H^{i+n}(E, E - \{0\}),\ y \longmapsto \pi^* y \smile u_E.$$

Proof. See Section 10, and in particular Theorem 10.4, of [MS74]. □

Now suppose that E contains a sub-bundle $E_{\mathbb{Z}}$ with fibers isomorphic to $\mathbb{Z}^n$. We can then construct the torus bundle $T := E/E_{\mathbb{Z}} \to X$. For every $x \in X$, the orientation on E_x yields an orientation on T_x. Fix $c \in \mathbb{Z}_{\geq 1}$ and consider the class

$$u_{T_x,c} \in H^n(T_x, T_x - T_x[c]) \simeq \bigoplus_{z \in T_x[c]} H^n(T_x, T_x - \{z\}),$$

which restricts to the generator of each $H^n(T_x, T_x - \{z\})$ determined by the orientation of T_x at z, for every $z \in T_x[c]$. By considering a tubular neighborhood of $T[c]$ in T, and applying the excision theorem, we deduce the Thom isomorphism for torus bundles from Theorem 2.1 above.

Theorem 2.2. *There is a unique class $u_{T,c} \in H^n(T, T - T[c])$ such that its pullback to any fiber T_x of T is equal to $u_{T_x,c}$. Moreover, for every $i \in \mathbb{Z}$, the Thom isomorphism in Theorem 2.1 induces an isomorphism*

$$H^i(T[c]) \xrightarrow{\sim} H^{i+n}(T, T - T[c]).$$

Definition 2.3. The class u_E is called the Thom class of the bundle $E \to X$, and $u_{T,c}$ is the Thom class of the torus bundle $T \to X$ relative to the c-torsion.

We now outline the definition of the Eisenstein class of the torus bundle $T \to X$ relative to the c-torsion. For this, we assume that for all $i \in \mathbb{Z}$, the group $H^i(X)$ is finitely generated. Consider the following class in singular cohomology

$$T[c] - c^n\{0\} \in H^0(T[c]).$$

Denote by the same symbol the image of this class in $H^n(T, T - T[c])$ via the Thom isomorphism given in Theorem 2.2. The long exact sequence in relative cohomology gives

$$\cdots \longrightarrow H^{n-1}(T) \longrightarrow H^{n-1}(T - T[c]) \longrightarrow H^n(T, T - T[c]) \longrightarrow H^n(T) \longrightarrow \cdots . \tag{3}$$

We then have the following theorem.

Theorem 2.4 (Sullivan, Bergeron–Charollois–García)**.** *There exists a unique class ${}_cz_T \in H^{n-1}(T - T[c], \mathbb{Z}[1/c])$ satisfying:*

(1) It is a lift of $T[c] - c^n\{0\} \in H^n(T, T - T[c], \mathbb{Z}[1/c])$ by the map in (3).
(2) It is invariant under pushforward induced by multiplication by a in T for all $a \in \mathbb{N}^{(c)}$.

Proof. Section 2 and Section 3 of [BCG20] prove the existence of the class ${}_cz_T$ with coefficients in $\mathbb{Z}[1/N]$, for N divisible by c and coprime to p (see the remarks below Lemma 9 and Definition 10 of [BCG20]). This is sufficient for our purposes, but we refer the reader to [Xu23, Page 14] for a proof that the coefficients can be taken to be $\mathbb{Z}[1/c]$. □

Definition 2.5. The class ${}_cz_T$ above is the Eisenstein class attached to T and c.

Throughout this work, we will define invariants attached to totally real fields of degree n from periods of Eisenstein classes of torus bundles of rank n.

Remark 2.6. Theorem 2.4 has the following visual interpretation. The first point is equivalent to the fact that the image of $T[c] - c^n\{0\}$ in $H^n(T, \mathbb{Z}[1/c])$ vanishes. Informally, this means that there is a codimension $n-1$ submanifold $\Sigma \subset T - T[c]$ such that

$$\partial\Sigma = t(T[c] - c^n\{0\}),\ t \in \mathbb{Z},$$

where $\partial\Sigma$ denotes the boundary of Σ. On the other hand, the class of Σ is not unique, and the second point of the theorem provides a preferred class, ${}_cz_T$ with this property. In particular, ${}_cz_T$ allows defining linking numbers with $T[c] - c^n\{0\}$ as the intersection number with the preferred choice of Σ.

Remark 2.7. Let $a \in \mathbb{N}^{(c)}$. Consider inclusion

$$i\colon T - T[ac] \hookrightarrow T - T[c].$$

Multiplication by a induces a map

$$[a]\colon T - T[ac] \longrightarrow T - T[c].$$

The map pushforward induced by multiplication by $[a]$ on $H^i(T - T[c])$ appearing in Theorem 2.4 is defined as the composition

$$H^i(T - T[c]) \xrightarrow{i^*} H^i(T - T[ac]) \xrightarrow{[a]_*} H^i(T - T[c]).$$

We similarly define $[a]_*\colon H^i(T, T - T[c]) \to H^i(T, T - T[c])$.

2.2. **Eisenstein class of universal families of tori.** Let $n \geq 2$ and denote by $\mathscr{X} := \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n$ the symmetric space attached to $\mathrm{SL}_n(\mathbb{R})$. We are interested in the Eisenstein class of universal families of tori over quotients of $\mathscr{X}$ by the following congruence subgroups.

Let p be an odd prime such that $(p, c) = 1$, for $r \geq 0$ consider the column vector

$$v_r := (1/p^r, 0, \ldots, 0)^t \in \mathbb{Q}^n/\mathbb{Z}^n,$$

and let Γ_r be its stabilizer in $\Gamma := \mathrm{SL}_n(\mathbb{Z})$. Fix $q \neq p$ an auxiliary prime such that the full level congruence subgroup $\Gamma(q) \subset \Gamma$ is torsion-free and has index prime to p. Observe that these conditions imply that p is sufficiently large. Finally, define $\Gamma_r(q) := \Gamma_r \cap \Gamma(q)$ and consider the torus bundle

$$T_r := \Gamma_r(q)\backslash(\mathscr{X} \times \mathbb{R}^n/\mathbb{Z}^n) \longrightarrow \Gamma_r(q)\backslash\mathscr{X}.$$

Definition 2.8. Denote by $z_r := {}_cz_{T_r} \in H^{n-1}(T_r - T_r[c], \mathbb{Z}[1/c])$ the Eisenstein class attached to the torus bundle T_r and c.

Remark 2.9. We introduced the auxiliary prime q and the congruence subgroups $\Gamma_r(q)$ to ensure that their action on $\mathscr{X}$ is free, which holds as $\Gamma_r(q)$ is torsion-free. Thus, the fibers of T_r are n-tori.

Remark 2.10. We are omitting c from the notation since it is generally fixed. We note in passing also that the dependence of our classes on c is simple: as explained in [BCG20, Section 3.3], it follows from the definition of the Eisenstein class that, if c and d are coprime,

$$([c]^* - c^n)_d z_r = ([d]^* - d^n)_c z_r.$$

For $r \geq 1$, the vector v_r induces a section

$$v_r \colon \Gamma_r(q)\backslash\mathscr{X} \longrightarrow T_r - T_r[c],\ [g] \longmapsto [(g, v_r)].$$

We can then consider the pullback $v_r^* z_r \in H^{n-1}(\Gamma_r(q)\backslash\mathscr{X}, \mathbb{Z}[1/c])$. We proceed to study the behavior of $v_r^* z_r$ with respect to two different actions. Observe that $\Gamma_r(q)$ is a normal subgroup of Γ_r. Thus, we can define an action of Γ_r on $\Gamma_r(q)\backslash\mathscr{X}$ as follows. For $\gamma \in \Gamma_r$,

$$\gamma \colon \Gamma_r(q)\backslash\mathscr{X} \longrightarrow \Gamma_r(q)\backslash\mathscr{X},\ [g] \longmapsto [\gamma g].$$

As a consequence, Γ_r acts on $H^{n-1}(\Gamma_r(q)\backslash\mathscr{X}, \mathbb{Z}[1/c])$ via pullback. Since Γ_r fixes v_r, we deduce that the class $v_r^* z_r$ is fixed by this action, as we make precise in the next lemma.

Lemma 2.11. *Consider the same notation as above. We have*

$$v_r^* z_r \in H^{n-1}(\Gamma_r(q)\backslash\mathscr{X}, \mathbb{Z}[1/c])^{\Gamma_r}.$$

Proof. Let $\gamma \in \Gamma_r$ and define the map of torus bundles

$$\tilde{\gamma} \colon T_r \longrightarrow T_r,\ [(g, v)] \longmapsto [(\gamma g, \gamma v)].$$

We have $\tilde{\gamma}^* T_r[c] = T_r[c]$ and $\tilde{\gamma}^*\{0\} = \{0\}$ in $H^0(T_r[c])$. Moreover, since $\tilde{\gamma}^* u_{T_r,c} = u_{T_r,c}$, as $\tilde{\gamma}$ is orientation preserving, it follows that

$$\tilde{\gamma}^*(T_r[c] - c^n\{0\}) = T_r[c] - c^n\{0\} \in H^n(T_r, T_r - T_r[c]).$$

This implies that $\tilde{\gamma}^* z_r$ is a lift of $T_r[c] - c^n\{0\}$. Moreover, for every $a \in \mathbb{N}^{(c)}$, $\tilde{\gamma}^*$ commutes with $[a]_*$. Indeed, define $\widetilde{\gamma^{-1}}$ in the same way as $\tilde{\gamma}$ but replacing γ by γ^{-1}. Since $\tilde{\gamma}^* = \widetilde{\gamma^{-1}}_*$, the desired commutativity follows then from taking the pushforward of $[a] \circ \widetilde{\gamma^{-1}} = \widetilde{\gamma^{-1}} \circ [a]$. From there, we deduce that $\tilde{\gamma}^* z_r$ is invariant under $[a]_*$. As a consequence, Theorem 2.4 implies $z_r = \tilde{\gamma}^* z_r$. Pulling back this equality by $v_r \colon \Gamma_r(q)\backslash\mathscr{X} \to T_r - T_r[c]$ yields the desired expression. □

Let $w = \mathrm{diag}(1, -1, 1, \ldots, 1) \in \mathrm{GL}_n(\mathbb{Z})$. Since w normalizes $\Gamma_r(q)$ and SO_n, conjugation induces the following map

$$w \colon \Gamma_r(q)\backslash\mathscr{X} \longrightarrow \Gamma_r(q)\backslash\mathscr{X},\ [g] \longmapsto [wgw^{-1}],$$

which induces an involution w on $H^{n-1}(\Gamma_r(q)\backslash\mathscr{X}, \mathbb{Z}[1/c])$ via pullback. Here and for the rest of the section, we will denote with a superindex $-$ the $w = -1$ eigenspace for w.

Lemma 2.12. *For every $r \geq 1$, we have*

$$v_r^* z_r \in H^{n-1}(\Gamma_r(q)\backslash\mathscr{X}, \mathbb{Z}[1/c])^-.$$

Proof. The proof is analogous to the proof of Lemma 2.11, so we only outline it. Denote by $\tilde{w}$ the morphism of torus bundles

$$\tilde{w} \colon T_r \longrightarrow T_r,\ [([g], v)] \longmapsto [(wgw^{-1}, wv)].$$

Since $\tilde{w}$ reverses the orientation on the fibers (because the determinant of the matrix defining w is -1), it follows that

$$\tilde{w}^*(T_r[c] - c^n\{0\}) = -\,(T_r[c] - c^n\{0\}) \in H^n(T_r, T_r - T_r[c]).$$

Similar to Lemma 2.11, we deduce from there that $\tilde{w}^* z_r = -z_r$. The desired result follows by pulling back this equality by v_r and observing $wv_r = v_r$. □

2.3. **Distribution relations.** We give some compatibility properties regarding the classes z_r and their pullbacks by torsion sections. In particular, we prove a distribution relation. We begin with the following general lemma.

Lemma 2.13. *Consider the commutative diagram of topological spaces, where all the maps are continuous*

$$\begin{array}{ccc} Z & \xrightarrow{f_1} & Y \\ \Big\downarrow{\scriptstyle h_1} & & \Big\downarrow{\scriptstyle h_2} \\ X & \xrightarrow{f_2} & S. \end{array}$$

Suppose that the following two conditions hold:

(1) h_1 and h_2 are r-sheeted covering maps, for $r \in \mathbb{Z}_{\geq 1}$.

(2) If $x \in X$ and $\{z_j\}_{j=1}^r$ are its distinct lifts by h_1, the images $\{f_1(z_j)\}_{j=1}^r$ are distinct.

Then, for all $i \in \mathbb{Z}_{\geq 0}$, we have

$$(h_1)_* f_1^* = f_2^* (h_2)_* \colon H^i(Y) \longrightarrow H^i(X).$$

Proof. For the proof of this lemma, we follow the same notation as in its statement. Let $\varphi \in C^i(Y, \mathbb{Z})$ be a degree i cochain and consider $\sigma\colon \Delta^i \to X$ a continuous map from an i-simplex Δ^i to X. Fix a vertex $u \in \Delta^i$, and let $x = \sigma(u)$.

Since h_1 is an r-sheeted covering map, there are $\tilde{\sigma}_1, \ldots, \tilde{\sigma}_r\colon \Delta^i \to Z$ distinct lifts of σ by h_1, characterized by the property $\tilde{\sigma}_j(u) = z_j$. Then,

$$(h_1)_* f_1^* \varphi(\sigma) = \sum_j f_1^* \varphi(\tilde{\sigma}_j) = \sum_j \varphi(f_1 \circ \tilde{\sigma}_j).$$

Similarly, let $y_1, \ldots, y_r \in Y$ be the distinct lifts of $f_2(x)$, and consider $\tilde{\omega}_1, \ldots, \tilde{\omega}_r\colon \Delta^i \to Y$ the distinct lifts of $f_2 \circ \sigma$ by h_2, characterized by the property $\tilde{\omega}_j(u) = y_j$. Then,

$$f_2^* (h_2)_* \varphi(\sigma) = (h_2)_* \varphi(f_2 \circ \sigma) = \sum_j \varphi(\tilde{\omega}_j).$$

We now observe that we have the equality of sets $\{f_1 \circ \tilde{\sigma}_j\}_j = \{\tilde{\omega}_j\}_j$. Indeed, Condition (2) in the statement of the lemma implies that the simplices $\{f_1 \circ \tilde{\sigma}_j\}_j$ are all distinct, which implies that both sets have the same number of elements. Moreover, since $f_1 \circ \tilde{\sigma}_j$ is a lift of $f_2 \circ \sigma$ by h_2, we deduce the inclusion $\{f_1 \circ \tilde{\sigma}_j\} \subset \{\tilde{\omega}_j\}_j$ and the desired equality of sets follows.

From this equality of sets and the previous two calculations, we obtain the desired equality $(h_1)_* f_1^* = f_2^* (h_2)_*$ of cochain maps, which induces the result in cohomology. □

Remark 2.14. Condition (2) of the lemma above holds if the commutative diagram is Cartesian.

Proposition 2.15. *Let $r, r' \in \mathbb{Z}$ with $r \geq r' \geq 1$, consider the projection map* $\mathrm{pr}\colon T_r - T_r[c] \to T_{r'} - T_{r'}[c]$, *and denote by* pr^* *the corresponding pullback in cohomology. Then,* $\mathrm{pr}^* z_{r'} = z_r$.

Proof. The structure of the proof is analogous to the proof of Lemma 2.11, so we only outline the key points. First, we observe

$$\mathrm{pr}^*(T_{r'}[c] - c^n\{0\}) = T_r[c] - c^n\{0\} \in H^n(T_r, T_r - T_r[c]).$$

Therefore, $\mathrm{pr}^*(z_{r'})$ is a lift of $T_r[c] - c^n\{0\}$. Second, we claim that pr^* commutes with $[a]_*$. The key to proving this statement is to apply Lemma 2.13 to the diagram

$$\begin{array}{ccc} T_r - T_r[ac] & \xrightarrow{\mathrm{pr}} & T_{r'} - T_{r'}[ac] \\ \big\downarrow{\scriptstyle [a]} & & \big\downarrow{\scriptstyle [a]} \\ T_r - T_r[c] & \xrightarrow{\mathrm{pr}} & T_{r'} - T_{r'}[c]. \end{array}$$

Therefore, $z_r = \mathrm{pr}^* z_{r'}$ by Theorem 2.4. □

From the previous proposition, we deduce that the classes $v_r^* z_r$ satisfy the following distribution relation.

Proposition 2.16. *Let $r \geq 1$ and consider the pushforward attached to the finite quotient map* $\mathrm{pr}\colon \Gamma_{r+1}(q)\backslash\mathscr{X} \to \Gamma_r(q)\backslash\mathscr{X}$*, namely*

$$\mathrm{pr}_*\colon H^{n-1}(\Gamma_{r+1}(q)\backslash\mathscr{X}, \mathbb{Z}[1/c]) \longrightarrow H^{n-1}(\Gamma_r(q)\backslash\mathscr{X}, \mathbb{Z}[1/c]).$$

Then, $\mathrm{pr}_*(v_{r+1}^* z_{r+1}) = v_r^* z_r$.

Proof. Consider the map

$$f_r\colon \Gamma_r(q)\backslash\mathscr{X} \longrightarrow T_r - T_r[c] \longrightarrow T_1 - T_1[c],$$

where the first arrow is induced by v_r and the second one is the quotient map. Also, observe that since $r \geq 1$ we can define

$$f_{r+1} : \Gamma_{r+1}(q)\backslash\mathscr{X} \longrightarrow T_{r+1} - T_{r+1}[pc] \longrightarrow T_1 - T_1[pc],$$

in a similar way as f_r, but where we used that v_{r+1} is of exact p^{r+1} torsion, with $p^{r+1} > p$. It is a consequence of Proposition 2.15 that, if $\iota\colon T_1 - T_1[pc] \to T_1 - T_1[c]$,

$$v_r^* z_r = f_r^* z_1, \quad v_{r+1}^* z_{r+1} = f_{r+1}^* \iota^* z_1.$$

We will now deduce the desired statement from the invariance of z_1 under multiplication by p. With this aim, observe that we can apply Lemma 2.13 to the following commutative diagram

$$\begin{array}{ccc} \Gamma_{r+1}(q)\backslash\mathscr{X} & \xhookrightarrow{f_{r+1}} & T_1 - T_1[pc] \\ \big\downarrow{\scriptstyle \mathrm{pr}} & & \big\downarrow{\scriptstyle [p]} \\ \Gamma_r(q)\backslash\mathscr{X} & \xhookrightarrow{f_r} & T_1 - T_1[c]. \end{array}$$

Indeed, since $\Gamma_r(q)$ is torsion-free and $[\Gamma_r(q) : \Gamma_{r+1}(q)] = p^n$, both horizontal maps are p^n-sheeted covering maps, implying Condition (1) of the lemma. Moreover, the fact that $\Gamma_r(q)$ is torsion-free implies that the maps f_r and f_{r+1} are injective, giving Condition (2) of the lemma. Therefore,

$$\mathrm{pr}_* f_{r+1}^* = f_r^* [p]_*.$$

From there,

$$\mathrm{pr}_* v_{r+1}^* z_{r+1} = \mathrm{pr}_* f_{r+1}^* \iota^* z_1 = f_r^* [p]_* \iota^* z_1 = f_r^* z_1 = v_r^* z_r,$$

where we used the invariance of z_1 under multiplication by p on the second to last equality (see Theorem 2.4 and Remark 2.7). □

3. Differential form representative of the Eisenstein class

In [BCG20], Bergeron, Charollois, and García construct a closed differential form on $T_r - T_r[c]$ representing the Eisenstein class z_r. Their construction, inspired by the work of Bismut and Cheeger [BC92], consists of a regularized average of a transgression form considered by Mathai and Quillen. In this section, we outline this procedure and use the differential forms we obtain to prove some properties about pullbacks of the Eisenstein class by torsion sections (see Proposition 3.10). The expressions given here will also be used in the last section to relate periods of the Eisenstein class to special values of L-functions.

3.1. **Mathai–Quillen form and the transgression form.** Let $S := \mathrm{GL}_n(\mathbb{R})/\mathrm{SO}_n$ and consider the real vector bundle $E := S \times \mathbb{R}^n \to S$, which is $\mathrm{GL}_n(\mathbb{R})$-equivariant for the left multiplication action on each of the components of E and on S. Mathai and Quillen construct a closed $\mathrm{GL}_n(\mathbb{R})$-equivariant differential form

$$\varphi \in \Omega^n_{\mathrm{rd}}(E)^{\mathrm{GL}_n(\mathbb{R})}$$

which has rapid decay (Gaussian shape) and integral 1 along the fibers. In particular, φ represents the Thom class of the oriented vector bundle $E \to S$ via the isomorphisms

$$H^n(\Omega^\bullet_{\mathrm{rd}}(E)) \simeq H^n(E, E - \{0\}, \mathbb{R})$$

between the cohomology of the complex of forms on E with rapid decay along the fibers $\Omega^\bullet_{\mathrm{rd}}(E)$ and relative singular cohomology (see [MQ86, Page 98 and Page 99]).

There is an explicit expression for the form φ, which we proceed to outline following [BCG20, Theorem 13] and [MQ86]. The reader is referred to these sources for further details on the construction of φ, as for our purposes it is sufficient to know the shape of its expression. Using the Iwasawa decomposition of $\mathrm{GL}_n(\mathbb{R})$, fix $h\colon S \to \mathrm{GL}_n(\mathbb{R})$ a smooth section of the quotient map $\mathrm{GL}_n(\mathbb{R}) \twoheadrightarrow S$. Then

$$\varphi = \pi^{-n/2} e^{-|h^{-1}x|^2} \sum_{\substack{I \subset \{1,\dots,n\} \\ |I| \text{ even}}} \varepsilon_{I,I'} \mathrm{Pf}(\Omega_I/2) \left(d(h^{-1}x) + \theta h^{-1}x\right)^{I'}, \tag{4}$$

where:

- $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $|x| = (x_1^2 + \cdots + x_n^2)^{1/2}$ is its standard norm.
- θ is an $n \times n$ matrix of 1-forms on S, obtained as the pullback by h of the connection of the principal SO_n-bundle $\mathrm{GL}_n(\mathbb{R}) \to S$ given by $\theta_{\mathrm{GL}_n(\mathbb{R})} = (g^{-1}dg - dg^t(g^t)^{-1})/2$.
- Ω is an $n \times n$ matrix of 2-forms on S, obtained as the pullback by h of the curvature $d\theta_{\mathrm{GL}_n(\mathbb{R})} + \theta^2_{\mathrm{GL}_n(\mathbb{R})}$. Then, $\mathrm{Pf}(\Omega_I/2)$ is an $|I|$-form given as the Pfaffian of the submatrix of $\Omega/2$ of size $|I|$ involving the indices in I.
- I' denotes the complement of $I \subset \{1, \dots, n\}$, $\varepsilon_{I,I'} \in \{\pm 1\}$, and for a vector v of size n, $v^{I'} = v_{i_1} v_{i_2} \cdots v_{i_{|I'|}}$, where $I' = \{i_1, \dots, i_{|I'|}\}$.

Remark 3.1. We will not use the expressions for θ and Ω, aside from the fact that they are forms of degree 1 and 2 on S.

For $t \in \mathbb{R}_{>0}$, let $[t]\colon E \to E$ be multiplication by t on the fibers. An important property of φ is that for every $t \in \mathbb{R}_{>0}$, $[t]^*\varphi$ also represents the Thom class. Indeed, the Gaussian on the fibers gets dilated, but the value of the integral over the fibers is preserved and equal to 1. In particular,

$$[t]^*\varphi \longrightarrow \delta_0, \quad \text{as } t \longrightarrow +\infty,$$

where δ_0 denotes the current of integration along the zero section of E, also represents the Thom class (as a current). Recall that the Eisenstein class is a lift of Thom classes of a torus bundle, by Theorem 2.4. The next proposition constructs a form η whose differential involves δ_0. The relevance of this form is that a (regularized) average of it will give a representative of the Eisenstein class.

Definition 3.2. Let $R := \sum_i x_i \frac{\partial}{\partial x_i}$ be the radial vector field on $E = S \times \mathbb{R}^n$, where $\{x_i\}_i$ denote the coordinates on $\mathbb{R}^n$ and consider the contraction $\psi := \iota_R \varphi \in \Omega^{n-1}_{\mathrm{rd}}(E)^{\mathrm{GL}_n(\mathbb{R})}$, which is $\mathrm{GL}_n(\mathbb{R})$-invariant (see [BCG20, Proposition 14]).

Proposition 3.3. *Consider the differential form on $E - S$*

$$\eta := \int_0^{+\infty} [t]^*\psi \frac{dt}{t}. \tag{5}$$

Viewed as a current on E, it satisfies the transgression property $d\eta = \delta_0 - [0]^*\varphi$.

Proof. The main idea for the proof of this statement lies in the following equalities

$$\delta_0 - [0]^*\varphi = \int_0^{+\infty} \frac{d}{dt}[t]^*\varphi dt = \int_0^{+\infty} d[t]^*\iota_R\varphi \frac{dt}{t} = d\eta,$$

where the second equality follows from interpreting $\frac{d}{dt}[t]^*\varphi$ in terms of a Lie derivative with respect to the vector field R and using Cartan magic formula. For more details, see Section 7.2 and Section 7.3 of [BCG20] and Page 106 of [MQ86]. □

Using the explicit expression for φ given in (4), and following the same notation as in that equation, we obtain

$$\psi = \pi^{-n/2} e^{-|h^{-1}x|^2} \sum_{\substack{I \subsetneq \{1,\dots,n\} \\ |I| \text{ even}}} \left(\varepsilon_{I,I'} \mathrm{Pf}(\Omega_I/2) \sum_{k=1}^{|I'|} (-1)^{k+1} (h^{-1}x)_{i_k} \left(d(h^{-1}x) + \theta h^{-1}x\right)^{I' - \{i_k\}} \right),$$

$$\eta = \frac{\pi^{-n/2}}{2} \sum_{\substack{I \subsetneq \{1,\dots,n\} \\ |I| \text{ even}}} \left(\varepsilon_{I,I'} \mathrm{Pf}(\Omega_I/2) \frac{\Gamma(|I'|/2)}{|h^{-1}x|^{|I'|}} \sum_{k=1}^{|I'|} (-1)^{k+1} (h^{-1}x)_{i_k} \left(d(h^{-1}x) + \theta h^{-1}x\right)^{I' - \{i_k\}} \right).$$

Here $I' = \{i_1, \dots, i_{|I'|}\}$ is the complement of $I \subsetneq \{1, \dots, n\}$. The exact formulas will not be necessary for us. On the other hand, it will be important to note:

- φ and ψ are linear combinations of products of an exponential and a polynomial. In particular, they have rapid decay along the fibers.
- $[0]^*\psi = 0$.
- η does not have rapid decay along the fibers.

3.2. **Eisenstein transgression.** We proceed to consider a regularized average of the form η in (5) over a lattice to obtain forms on torus bundles representing the Eisenstein class. For $L \subset \mathbb{Q}^n$ a $\mathbb{Z}$-lattice and $\lambda \in L$, let

$$\mathrm{tr}_\lambda \colon E \longrightarrow E,\ (g, x) \longmapsto (g, x + \lambda).$$

Then, if $t \in \mathbb{R}_{>0}$, define

$$\theta([t]^*\psi, L) := \sum_{\lambda \in L} \mathrm{tr}_\lambda^* [t]^* \psi. \tag{6}$$

The sum converges as the differential form $t^*\psi$ has rapid decay on the fibers of $E \to S$.

Theorem 3.4. *View $\theta([t]^*\psi, L)$ as a differential form on $S \times (\mathbb{R}^n - L)$. For $s \in \mathbb{C}$ with $\mathrm{Re}(s) \gg 0$, the integral*

$$E_\psi(L, s) := \int_0^{+\infty} \theta([t]^*\psi, L) t^s \frac{dt}{t}$$

converges. Furthermore, it admits a meromorphic continuation to all $s \in \mathbb{C}$, regular at $s = 0$, and its value at every regular $s \in \mathbb{C}$ defines a differential form on $S \times (\mathbb{R}^n - L)$.

Proof. This follows from Proposition 17 and Section 8.5 of [BCG20]. In particular, the fact that the integral is regular at $s = 0$ follows from the fact that we are viewing $\theta([t]^*\psi, L)$ as a form on $S \times (\mathbb{R}^n - L)$, and $[t]^*\psi$ tends to 0 as $t \to +\infty$ on $S \times (\mathbb{R}^n - L)$. □

The previous theorem implies that $E_\psi(L, s)$ is regular at $s = 0$ and

$$E_\psi(L) := E_\psi(L, 0)$$

defines a form on $S \times (\mathbb{R}^n - L)/L$. In fact, $E_\psi(L)$ descends to a form in $\mathscr{X} \times (\mathbb{R}^n - L)/L$ by the calculation on (8.9) of [BCG20]. Moreover, if $\Gamma' \subset \mathrm{SL}_n(\mathbb{R})$ is a subgroup contained in the stabilizer of L, the form $E_\psi(L)$ is invariant under Γ'.

Remark 3.5. We outline how to view $E_\psi(L)$ as a regularized average of η. As we pointed out at the end of Section 3.1, the form η does not have rapid decay along the fibers. Therefore, the sum $\sum_{\lambda \in L} \mathrm{tr}_\lambda^* \eta$ does not converge. On the other hand, for $s \in \mathbb{C}$ with $\mathrm{Re}(s) \gg 0$ define

$$\eta(s) := \int_0^{+\infty} [t]^*\psi t^s \frac{dt}{t}.$$

Then, $\eta(s)$ has the same expression as the one given for η at the end of Section 3.1 where the term $\Gamma(|I'|/2)/|h^{-1}x|^{|I'|}$ is replaced by $\Gamma((|I'| + s)/2)/|h^{-1}x|^{|I'|+s}$. In particular, it follows that if $\mathrm{Re}(s) \gg 0$, the sum $\sum_{\lambda \in L} \mathrm{tr}_\lambda^* \eta(s)$ is absolutely convergent and

$$E_\psi(L, s) = \int_0^{+\infty} \theta([t]^*\psi, L) t^s \frac{dt}{t} = \sum_{\lambda \in L} \mathrm{tr}_\lambda^* \eta(s),$$

where we exchanged the integral with the sum (using that for $\mathrm{Re}(s) \gg 0$, the sums are absolutely convergent). Thus, $E_\psi(L)$ is equal to the value at $s = 0$ of the meromorphic continuation of $\sum_{\lambda \in L} \mathrm{tr}_\lambda^* \eta(s)$.

Recall the torus bundle

$$T_r = \Gamma_r(q) \backslash (\mathscr{X} \times \mathbb{R}^n/\mathbb{Z}^n) \longrightarrow \Gamma_r(q) \backslash \mathscr{X}$$

introduced in Section 2.2.

Definition 3.6. Consider the linear combination

$$ {}_cE_\psi := E_\psi(c^{-1}\mathbb{Z}^n) - c^n E_\psi(\mathbb{Z}^n), $$

which we view as a differential form on $T_r - T_r[c]$ for every $r \geq 1$.

Theorem 3.7. *The form ${}_cE_\psi$ is closed in $T_r - T_r[c]$ and its cohomology class*

$$ {}_cE_\psi \in H^{n-1}_{\mathrm{dR}}(T_r - T_r[c]) \simeq H^{n-1}(T_r - T_r[c], \mathbb{R}) $$

is equal to the image of the Eisenstein class z_r in $H^{n-1}(T_r - T_r[c], \mathbb{R})$.

Proof. See Theorem 19, Proposition 20, and Theorem 21 of [BCG20]. There it is explained that, since E_ψ is a regularized average of η (see Remark 3.5), Proposition 3.3 implies that

$$ d({}_cE_\psi) = \delta_{T[c]} - c^n \delta_{\{0\}}, $$

where $\delta_{T[c]}$ and $\delta_{\{0\}}$ denote currents of integration along $T[c]$ and $\{0\}$ (the contributions $[0]^*\varphi$ appearing in Proposition 3.3 vanish after the regularization). Moreover, $[a]_*E_\psi = E_\psi$ by Proposition 20 of [BCG20]. Thus, ${}_cE_\psi$ is a closed form on $T_r - T_r[c]$ satisfying the characterizing properties of the Eisenstein class z_r asserted in Theorem 2.4. □

3.3. **Pullbacks by torsion sections.** We now use the differential forms introduced above to study the pullbacks of the form ${}_cE_\psi$ by torsion sections. For $v \in \mathbb{Q}^n$, denote also by v the corresponding section $v\colon S \to E$. Then, for $s \in \mathbb{C}$ with $\mathrm{Re}(s) \gg 0$, consider the differential form on S

$$ \eta(v,s) := \int_0^{+\infty} v^*[t]^*\psi t^s \frac{dt}{t} = \int_0^{+\infty} (tv)^*\psi t^s \frac{dt}{t}. $$

Since $0^*\psi = 0$, which can be verified using the explicit expression given at the end of Section 3.1, we have $\eta(0,s) = 0$. From this same expression and Remark 3.5, we deduce that for $v \neq 0$

$$ \eta(v,s) = $$

$$ \frac{\pi^{-n/2}}{2} \sum_{\substack{I \subsetneq \{1,\dots,n\} \\ |I| \text{ even}}} \left(\varepsilon_{I,I'} \mathrm{Pf}(\Omega_I/2) \frac{\Gamma((|I'|+s)/2)}{|h^{-1}v|^{|I'|+s}} \sum_{k=1}^{|I'|} (-1)^{k+1} (h^{-1}v)_{i_k} \left(d(h^{-1})v + \theta h^{-1}v\right)^{I'-\{i_k\}} \right). \tag{7} $$

Proposition 3.8. *Let $L \subset \mathbb{Q}^n$ be a $\mathbb{Z}$-lattice and $v \in \mathbb{Q}^n - L$. For $s \in \mathbb{C}$ with* $\mathrm{Re}(s) \gg 0$,

$$ v^*E_\psi(L,s) = \sum_{\lambda \in v+L} \eta(\lambda, s). $$

In particular, the right-hand side has a meromorphic continuation regular at $s = 0$.

Proof. This follows from Theorem 3.4 and Remark 3.5. □

Thus, if $v \in \mathbb{Q}^n - (1/c)\mathbb{Z}^n$,

$$ v^*{}_cE_\psi = \lim_{s \to 0} \sum_{\lambda \in v + c^{-1}\mathbb{Z}^n} \eta(\lambda, s) - c^n \sum_{\lambda \in v+\mathbb{Z}^n} \eta(\lambda, s), \tag{8} $$

where here and from now on, $\lim_{s\to 0}$ denotes evaluation of the meromorphic continuation.

In fact, the right-hand side of the equation appearing in Proposition 3.8 defines a differential form on S even if $v \in L$. More precisely,

$$\sum_{\lambda \in L} \eta(\lambda, s)$$

converges for $\mathrm{Re}(s) \gg 0$, and admits a meromorphic continuation to $\mathbb{C}$ which is regular at $s = 0$. We proceed to prove a weaker version of this statement, as this will be enough for our purposes.

Lemma 3.9. *Let $g \in S$, consider tangent vectors $Y_1, \dots, Y_{n-1} \in T_g S$ and denote $Y = (Y_1, \dots, Y_{n-1})$. Then, for $s \in \mathbb{C}$ with $\mathrm{Re}(s) \gg 0$,*

$$s \longmapsto \sum_{\lambda \in L} \eta(\lambda, s)_g(Y)$$

converges and admits a meromorphic continuation to $\mathbb{C}$ which is regular at $s = 0$.

Proof. It follows from the explicit expression of $\eta(v, s)$ given in (7) that the sum

$$\sum_{\lambda \in L} \eta(\lambda, s)_g(Y)$$

is absolutely convergent for $\mathrm{Re}(s) \gg 0$. From there, we deduce that if $\mathrm{Re}(s) \gg 0$, we have the equality

$$\sum_{\lambda \in L} \eta(\lambda, s)_g(Y) = \int_0^{+\infty} \sum_{\lambda \in L} \left((t\lambda)^* \psi\right)_g (Y) t^s \frac{dt}{t},$$

as we can exchange the integral with the sum. Thus, it is enough to prove that the right-hand side has a meromorphic continuation regular at $s = 0$. For that, define the function

$$f \colon \mathbb{R}^n \longrightarrow \mathbb{R},\ v \longmapsto (v^* \psi)_g(Y).$$

Since ψ is a differential form which has rapid decay along the fibers, it follows that f is a Schwartz function. Hence, we need to prove that

$$\int_0^{+\infty} \sum_{\lambda \in L} f(t\lambda) t^s \frac{dt}{t} \tag{9}$$

has a meromorphic continuation to $s \in \mathbb{C}$ which is regular at $s = 0$. We split the integral as a sum of integrals from 1 to $+\infty$ and from 0 to 1. Observe that $f(0) = 0$, as $0^* \psi = 0$. The rapid decay of f, together with the fact that $f(0) = 0$, implies that the integral from 1 to $+\infty$ converges absolutely and defines an entire function on s. To study the integral from 0 to 1, we use Poisson summation formula

$$\int_0^1 \sum_{\lambda \in L} f(t\lambda) t^s \frac{dt}{t} = \int_0^1 \sum_{\lambda \in L^\vee} \hat{f}(\lambda / t) t^{s-n} \frac{dt}{t},$$

where $\hat{f}$ denotes the Fourier transform of f and $L^\vee$ the dual lattice of L. For $\mathrm{Re}(s) \gg n$, the previous integral can be written as

$$\frac{\hat{f}(0)}{s-n} + \int_1^{+\infty} \sum_{\lambda \in L^\vee - \{0\}} \hat{f}(\lambda u) u^{n-s} \frac{du}{u}.$$

Since $\hat{f}$ is a Schwartz function, the integral converges for all values of $s \in \mathbb{C}$ and defines an entire function. Thus, this expression gives a meromorphic continuation of the integral from 0 to 1 regular everywhere except maybe at $s = n$. The result follows from there. □

Finally, we are ready to prove the following expression regarding pullbacks of the Eisenstein class by torsion sections, which will be useful for the next section.

Proposition 3.10. *For $v \in \mathbb{Q}^n - c^{-1}\mathbb{Z}^n$, view $v^*{}_cE_\psi$ as a differential form on $\mathscr{X}$. Then,*

$$\sum_{v \in \frac{1}{p}\mathbb{Z}^n/\mathbb{Z}^n - \{0\}} v^*{}_cE_\psi = 0.$$

Proof. By Proposition 3.8, and more precisely (8), we can write the sum of the proposition as the evaluation at $s = 0$ of the following expression

$$\sum_{v \in \frac{1}{p}\mathbb{Z}^n/\mathbb{Z}^n - \{0\}} \sum_{\lambda \in v + c^{-1}\mathbb{Z}^n} \eta(\lambda, s) - c^n \sum_{v \in \frac{1}{p}\mathbb{Z}^n/\mathbb{Z}^n - \{0\}} \sum_{\lambda \in v + \mathbb{Z}^n} \eta(\lambda, s).$$

We will verify that each of the two terms vanishes when evaluated at $s = 0$. Since the proof is analogous in the two cases, we will show that

$$\lim_{s \to 0} \sum_{v \in \frac{1}{p}\mathbb{Z}^n/\mathbb{Z}^n - \{0\}} \sum_{\lambda \in v + \mathbb{Z}^n} \eta(\lambda, s) = 0.$$

Let $g \in S$, consider tangent vectors $Y_1, \ldots, Y_{n-1} \in T_gS$, and let $Y = (Y_1, \ldots, Y_{n-1})$. Then, it is enough to see

$$\lim_{s \to 0} \sum_{v \in \frac{1}{p}\mathbb{Z}^n/\mathbb{Z}^n - \{0\}} \sum_{\lambda \in v + \mathbb{Z}^n} \eta(\lambda, s)_g(Y) = 0.$$

Then, for $s \in \mathbb{C}$ with $\mathrm{Re}(s) \gg 0$

$$\sum_{v \in \frac{1}{p}\mathbb{Z}^n/\mathbb{Z}^n - \{0\}} \sum_{\lambda \in v + \mathbb{Z}^n} \eta(\lambda, s)_g(Y) = \sum_{v \in \frac{1}{p}\mathbb{Z}^n} \eta(\lambda, s)_g(Y) - \sum_{\lambda \in \mathbb{Z}^n} \eta(\lambda, s)_g(Y),$$

where the right-hand side consists of the difference of two functions which admit a meromorphic continuation to all $s \in \mathbb{C}$ and are regular at $s = 0$ by Lemma 3.9. The previous expression is equal to

$$\sum_{\lambda \in \mathbb{Z}^n} \eta(\lambda/p, s)_g(Y) - \sum_{\lambda \in \mathbb{Z}^n} \eta(\lambda, s)_g(Y) = p^s \sum_{\lambda \in \mathbb{Z}^n} \eta(\lambda, s)_g(Y) - \sum_{\lambda \in \mathbb{Z}^n} \eta(\lambda, s)_g(Y).$$

Here we used that $\eta(\lambda/p, s) = p^s \eta(\lambda, s)$, which can be verified from the definition of $\eta(v, s)$. Since the meromorphic continuation of $\sum_{\lambda \in \mathbb{Z}^n} \eta(\lambda, s)_g(Y)$ is regular at $s = 0$ by Lemma 3.9, the evaluation at $s = 0$ of the expression above is zero. □

4. The Eisenstein group cohomology class

In this section, we package the pullbacks of the Eisenstein class by p-power torsion sections in a group cohomology class for $\Gamma := \mathrm{SL}_n(\mathbb{Z})$ valued in measures on $\mathbb{X} := \mathbb{Z}_p^n - p\mathbb{Z}_p^n$. Then, we discuss the process of lifting this class to a class valued in total mass zero measures on $\mathbb{X}$, which will be an important property for defining rigid classes and p-adic invariants attached to totally real fields.

4.1. **From singular to group cohomology.** Let $r \geq 1$ and let

$$m := \operatorname{lcm}(c, [\Gamma : \Gamma(q)], 2).$$

Since $\Gamma_r(q)$ is normal in Γ_r, there are actions of Γ_r on the singular cohomology of $\Gamma_r(q)\backslash\mathscr{X}$, described above Lemma 2.11, and on the group cohomology of $\Gamma_r(q)$, via conjugation. These actions are compatible with the natural isomorphism from singular to group cohomology, giving

$$H^{n-1}(\Gamma_r(q)\backslash\mathscr{X}, \mathbb{Z}[1/m])^{\Gamma_r} \xrightarrow{\sim} H^{n-1}(\Gamma_r(q), \mathbb{Z}[1/m])^{\Gamma_r} \xrightarrow{\sim} H^{n-1}(\Gamma_r, \mathbb{Z}[1/m]). \tag{10}$$

The first map is an isomorphism because $\Gamma_r(q)$ acts freely on $\mathscr{X}$, as it is torsion-free. The second one is given by the corestriction map multiplied by $[\Gamma_r : \Gamma_r(q)]^{-1}$, which belongs to $\mathbb{Z}[1/m]$ as $[\Gamma_r : \Gamma_r(q)]$ divides $[\Gamma : \Gamma(q)]$. The inverse of the second map is restriction.

For every $r \geq 1$, in Section 2.2 we constructed the classes

$$v_r^* z_r \in H^{n-1}(\Gamma_r(q)\backslash\mathscr{X}, \mathbb{Z}[1/m])^{\Gamma_r,-}.$$

and proved they are invariant under the action of Γ_r and belong to the -1-eigenspace for the action induced by $w := \operatorname{diag}(1, -1, 1, \ldots, 1) \in \mathrm{GL}_n(\mathbb{Z})$ in Lemma 2.11 and Lemma 2.12.

Definition 4.1. For $r \geq 1$, let $c_r \in H^{n-1}(\Gamma_r, \mathbb{Z}[1/m])$ be the group cohomology class corresponding to $v_r^* z_r$ via the isomorphisms of (10).

Similarly as above, $w \in \mathrm{GL}_n(\mathbb{Z})$ induces an action on group cohomology for $\Gamma_r(q)$ (as well as for Γ_r) via conjugation. This is compatible with the involution in singular cohomology induced by w considered in Section 2.2 via (10). It follows that $c_r \in H^{n-1}(\Gamma_r, \mathbb{Z}[1/m])^-$, where here and form now on the superindex $-$ indicates the -1-eigenspace for w.

The trace compatibility of the singular cohomology classes $(v_r^* z_r)_r$ leads to the compatibility of the group cohomology classes $(c_r)_r$ with respect to corestriction maps.

Proposition 4.2. *For $r \geq 1$ let $\operatorname{cor}\colon H^{n-1}(\Gamma_{r+1}, \mathbb{Z}[1/m]) \to H^{n-1}(\Gamma_r, \mathbb{Z}[1/m])$ be the corestriction map. Then, $\operatorname{cor}(c_{r+1}) = c_r$.*

Proof. Denote by $c_r(q) \in H^{n-1}(\Gamma_r(q), \mathbb{Z}[1/m])^{\Gamma_r}$ the image of $v_r^* z_r$ via the first isomorphism in (10). Since this isomorphism is compatible with respect to pushforward and corestriction (see [Bro82, Chapter III, Section 9 (E)]), it follows from Proposition 2.16 that if

$$\operatorname{cor}_q\colon H^{n-1}(\Gamma_{r+1}(q), \mathbb{Z}[1/m]) \longrightarrow H^{n-1}(\Gamma_r(q), \mathbb{Z}[1/m]),$$

denotes corestriction in group cohomology, then $\operatorname{cor}_q(c_{r+1}(q)) = c_r(q)$. This implies that

$$\operatorname{cor}\left([\Gamma_{r+1} : \Gamma_{r+1}(q)]c_{r+1}\right) = [\Gamma_r : \Gamma_r(q)]c_r.$$

which leads to the desired result as $[\Gamma_{r+1} : \Gamma_{r+1}(q)] = [\Gamma_r : \Gamma_r(q)] \in \mathbb{Z}[1/m]^\times$. $\square$

It is a computation to verify that the corestriction maps are equivariant with respect to the involution w. From there, we conclude

$$(c_r)_r \in \varprojlim_r H^{n-1}(\Gamma_r, \mathbb{Z}[1/m])^-.$$

4.2. **Cohomology class with coefficients in $\mathbb{Z}[1/m]$-measures.** We first describe the action of $w \in \mathrm{GL}_n(\mathbb{Z})$ on group cohomology with coefficients. For every $r \geq 0$ (including $\Gamma_0 = \Gamma$), conjugation by w induces the automorphism $\alpha\colon \Gamma_r \to \Gamma_r$, $\alpha(\gamma) = w\gamma w$. Then, if M is a $\mathrm{GL}_n(\mathbb{Z})$-module M, we can consider the morphism of complexes of group cochains

$$C^\bullet(\Gamma_r, M) \longrightarrow C^\bullet(\Gamma_r, M),\ c \longmapsto w \circ c \circ \alpha^r,$$

which induces an involution w on $H^i(\Gamma_r, M)$. We will denote by $H^i(\Gamma_r, M)^-$ the (-1)-eigenspace for w.

For $r \geq 1$, let $\mathbb{X}_r := (\mathbb{Z}/p^r\mathbb{Z})^n - (p\mathbb{Z}/p^r\mathbb{Z})^n$ and if A is an abelian group, denote

$$\mathbb{D}(\mathbb{X}_r, A) := \mathrm{Maps}(\mathbb{X}_r, A).$$

It admits a left action of $\mathrm{GL}_n(\mathbb{Z})$ given by $(g \cdot \lambda)(x) = \lambda(g^{-1}x)$, for $g \in \mathrm{GL}_n(\mathbb{Z})$, $\lambda \in \mathbb{D}(\mathbb{X}_r, A)$, and $x \in \mathbb{X}_r$. Let $x_r := (1, 0, \ldots, 0)^t \in \mathbb{X}_r$. Since the stabilizer of x_r in Γ is Γ_r, we deduce that we have a Γ-equivariant isomorphism

$$\mathrm{coInd}_{\Gamma_r}^{\Gamma}(A) \xrightarrow{\sim} \mathbb{D}(\mathbb{X}_r, A),\ f \longmapsto \lambda_f,$$

where $\lambda_f(x) = f(\gamma)$ for $\gamma \in \Gamma$ such that $\gamma x_r = x$. In particular, Shapiro's lemma induces an isomorphism

$$H^i(\Gamma, \mathbb{D}(\mathbb{X}_r, \mathbb{Z}[1/m]) \xrightarrow{\sim} H^i(\Gamma_r, \mathbb{Z}[1/m]),\ [\lambda] \longmapsto [c(\lambda)] \tag{11}$$

where $c(\lambda)(\gamma_0, \ldots, \gamma_i) = \lambda(\gamma_0, \ldots, \gamma_i)(x_r)$. Moreover, the isomorphism is equivariant with respect to the action of w.

Definition 4.3. For every $r \geq 1$, define $\mu_r \in H^{n-1}(\Gamma, \mathbb{D}(\mathbb{X}_r, \mathbb{Z}[1/m]))^-$ to be the image of c_r by the inverse of the isomorphism induced by Shapiro's lemma given in (11).

Consider the $\mathrm{GL}_n(\mathbb{Z})$-equivariant maps

$$u_{r+1}\colon \mathbb{D}(\mathbb{X}_{r+1}, A) \longrightarrow \mathbb{D}(\mathbb{X}_r, A),\ u_{r+1}(f)(x) = \sum_{\substack{x' \in \mathbb{X}_{r+1} \\ x' \equiv x \mod p^r}} f(x'). \tag{12}$$

It follows from the compatibility of the classes $(c_r)_r \in \varprojlim_r H^{n-1}(\Gamma_r, \mathbb{Z}[1/m])^-$ proven in Proposition 4.2, that we have a compatible system

$$(\mu_r)_r \in \varprojlim_r H^{n-1}(\Gamma, \mathbb{D}(\mathbb{X}_r, \mathbb{Z}[1/m]))^-,$$

where the transition maps are given by u_r for every $r \geq 2$. This statement can be proven using Chapter III, Section 9 (A) of [Bro82], which leads to describing the corestriction maps in terms of the map given by Shapiro's lemma and u_r.

Denote by $\mathbb{D}(\mathbb{X}, A)$ the space of A-valued distributions on $\mathbb{X}$. An element of $\lambda \in \mathbb{D}(\mathbb{X}, A)$ is determined by the values $\lambda(U)$ of the characteristic functions of compact open sets U. In particular, it is determined by the images of the following compact open sets. For $x \in \mathbb{X}_r$, choose any lift of it in $\mathbb{X}$, also denoted by x, and let

$$U_{x/p^r} := x + p^r\mathbb{Z}_p^n \subset \mathbb{X}. \tag{13}$$

Endow $\mathbb{D}(\mathbb{X}, A)$ with a left action of $\mathrm{GL}_n(\mathbb{Z})$ given by $(g \cdot \lambda)(U) = \lambda(g^{-1}U)$ and define

$$\mathbb{D}(\mathbb{X}, A) \twoheadrightarrow \mathbb{D}(\mathbb{X}_r, A),\ \lambda \longmapsto \lambda_r,$$

where for $x \in \mathbb{X}_r$, $\lambda_r(x) = \lambda(U_{x/p^r})$. This discussion implies the following lemma.

Lemma 4.4. *Let A be an abelian group. The map*

$$\mathbb{D}(\mathbb{X}, A) \xrightarrow{\sim} \varprojlim_r \mathbb{D}(\mathbb{X}_r, A), \ \lambda \longmapsto (\lambda_r)_r,$$

is a Γ-equivariant isomorphism.

We will now combine the compatible system of classes $(\mu_r)_r$ to a group cohomology class valued on $\mathbb{D}(\mathbb{X}, \mathbb{Z}[1/m])$. First, we note the following fact regarding the cohomology of $\Gamma = \Gamma_0$ and the congruence subgroups Γ_r for $r \geq 1$ in the stable range, which is a consequence of the work of Li and Sun [LS19].

Lemma 4.5. *For every $r \geq 0$ and $0 \leq i \leq n-2$, the group $H^i(\Gamma_r, \mathbb{Z}[1/m])^-$ is finite.*

Proof. Denote by $\tilde{\Gamma}_r$ the stabilizer of $v_r \in \mathbb{Q}^n/\mathbb{Z}^n$ in $\mathrm{GL}_n(\mathbb{Z})$. Then, Shapiro's lemma implies the following isomorphisms.

$$H^i(\Gamma_r, \mathbb{R})^- \simeq H^i(\tilde{\Gamma}_r, \mathbb{R}(\det)) \simeq H^i(\mathrm{GL}_n(\mathbb{Z}), I),$$

where $I := \mathrm{coInd}_{\tilde{\Gamma}_r}^{\mathrm{GL}_n(\mathbb{Z})}(\mathbb{R}(\det))$. Since $I^{\mathrm{GL}_n(\mathbb{Z})} = 0$, again by Shapiro's lemma, it follows from Example 1.10 of [LS19] that

$$H^i(\Gamma_r, \mathbb{R})^- = 0.$$

It is now an application of the universal coefficient theorem that $H^i(\Gamma_r, \mathbb{Z}[1/m])^-$ is torsion.

By [BS73, Theorem 11.4], the group Γ_r is of type (WFL). In particular, it is of type (VFL). By the Remark in Page 101 of Section 1.8 of [Ser71], and the universal coefficient theorem, it follows that $H^i(\Gamma_r, \mathbb{Z}[1/m])^-$ is finitely generated over $\mathbb{Z}[1/m]$. Since it is also a torsion group, we deduce that it is finite, as desired. □

Proposition 4.6. *For every $0 \leq i \leq n-1$, the map $\lambda \mapsto (\lambda_r)_r$ of Lemma 4.4 induces an isomorphism*

$$H^i(\Gamma, \mathbb{D}(\mathbb{X}, \mathbb{Z}[1/m]))^- \xrightarrow{\sim} \varprojlim_r H^i(\Gamma, \mathbb{D}(\mathbb{X}_r, \mathbb{Z}[1/m]))^-.$$

Proof. To simplify the notation, denote $\mathbb{D} := \mathbb{D}(\mathbb{X}, \mathbb{Z}[1/m])$ and $\mathbb{D}_r := \mathbb{D}(\mathbb{X}_r, \mathbb{Z}[1/m])$. For a group G, a G-module M, and $j \in \mathbb{Z}_{\geq 0}$, let

$$C^j(G, M) := \mathrm{Hom}_G(\mathbb{Z}[G^{j+1}], M),$$

where the action of G in G^{j+1} is diagonal. The complex $C^\bullet(G, M)$ with the usual coboundary maps computes the group cohomology of G with coefficients in M.

The surjective morphisms $u_r \mod p^{r-1} \colon \mathbb{D}_r \to \mathbb{D}_{r-1}$ obtained by taking the maps in (12) modulo p^{r-1} induce a map

$$u = (u_r) \colon \prod_{r \geq 1} C^i(\Gamma, \mathbb{D}_r) \twoheadrightarrow \prod_{r \geq 1} C^i(\Gamma, \mathbb{D}_r).$$

Since u_r is surjective for every r, u is surjective. It can be deduced from there and the expression of u, that $1 - u$ is also surjective, where 1 denotes the identity. In particular, we have a short exact sequence of complexes

$$0 \longrightarrow C^\bullet(\Gamma, \mathbb{D}) \longrightarrow \prod_{r\geq 1} C^\bullet(\Gamma, \mathbb{D}_r) \xrightarrow{1-u} \prod_{r\geq 1} C^\bullet(\Gamma, \mathbb{D}_r) \longrightarrow 0.$$

Note that to justify exactness in the middle, we used that $\mathbb{D} = \varprojlim_r \mathbb{D}_r$ by Lemma 4.4. Since 2 is invertible in $\mathbb{Z}[1/m]$, we can consider the $w = -1$ eigenspace of the corresponding long exact sequence in cohomology. This yields to the short exact sequence

$$0 \longrightarrow R^1 \varprojlim_r H^{i-1}(\Gamma, \mathbb{D}_r)^- \longrightarrow H^i(\Gamma, \mathbb{D})^- \longrightarrow \varprojlim_r H^i(\Gamma, \mathbb{D}_r)^- \longrightarrow 0, \tag{14}$$

where we used that

$$R^1 \varprojlim_r H^{i-1}(\Gamma, \mathbb{D}_r)^- = \operatorname{coker}\left(\prod_{r\geq 1} H^{i-1}(\Gamma, \mathbb{D}_r)^- \xrightarrow{1-u} \prod_{r\geq 1} H^{i-1}(\Gamma, \mathbb{D}_r)^-\right).$$

Finally, since $H^{i-1}(\Gamma, \mathbb{D}_r)^- \simeq H^{i-1}(\Gamma_r, \mathbb{Z}[1/m])^-$ is finite for $i-1 \leq n-2$ by Lemma 4.5, it follows that $\left(H^{i-1}(\Gamma, \mathbb{D}_r)^-\right)_r$ satisfies the Mittag–Leffler condition. Thus, $R^1 \varprojlim_r H^{i-1}(\Gamma, \mathbb{D}_r)^- = 0$ for every $i - 1 \leq n - 2$ proving the desired isomorphism. □

Definition 4.7. Define

$$\mu \in H^{n-1}(\Gamma, \mathbb{D}(\mathbb{X}, \mathbb{Z}[1/m]))^-$$

to be the class corresponding to $(\mu_r)_r$ via the isomorphism of Proposition 4.6.

Remark 4.8. The class μ viewed as a class with coefficients in $\mathbb{Z}_p$-valued measures on $\mathbb{X}$ is equal to the restriction of the classes considered in [BKL18, Definition 1.8.4] to measures on primitive vectors on $\mathbb{Z}_p^n$.

4.3. **Cocycle with coefficients in $\mathbb{R}$-distributions.** Using the differential form ${}_cE_\psi$ introduced in Section 3, which represents the Eisenstein class, we give an explicit representative of the image of $\mu_r \in H^{n-1}(\Gamma, \mathbb{D}(\mathbb{X}_r, \mathbb{Z}[1/m]))$ in $H^{n-1}(\Gamma, \mathbb{D}(\mathbb{X}_r, \mathbb{R}))$. This will be used to lift μ to a class valued in measures of total mass zero and to compare our constructions to special values of L-functions.

Lemma 4.9. *Let $r \geq 1$ and let $z \in \mathscr{X}$ be an arbitrary point. The map*

$$c_{v_r^* {}_cE_\psi} : \Gamma_r^n \longrightarrow \mathbb{R},\ (\gamma_0, \ldots, \gamma_{n-1}) \longmapsto \int_{\Delta(\gamma_0 z, \ldots, \gamma_{n-1} z)} v_r^* {}_cE_\psi,$$

where $\Delta(\gamma_0 z, \ldots, \gamma_{n-1} z)$ denotes the geodesic simplex in $\mathscr{X}$ with vertices $\{\gamma_i z\}_i$, defines a group cocycle and represents the class $c_r \in H^{n-1}(\Gamma_r, \mathbb{R})$.

Proof. The form $v_r^* {}_cE_\psi$ on $\mathscr{X}$ is closed and invariant under the action of Γ_r. It follows from there that $c_{v_r^* {}_cE_\psi}$ is a group cocycle and its cohomology class is independent of the choice of point $z \in \mathscr{X}$.

We proceed to see that the class of $c_{v_r^* {}_cE_\psi}$ is c_r. For this, note that Theorem 3.7 implies that $v_r^* {}_cE_\psi$ descends to a closed differential form on $\Gamma_r(q)\backslash\mathscr{X}$ representing $v_r^* z_r \in H^{n-1}(\Gamma_r(q)\backslash\mathscr{X}, \mathbb{R})$. Thus, the image of $v_r^* z_r$ by the first map in the isomorphism (10) (with coefficients in $\mathbb{R}$) is represented by the restriction of $c_{v_r^* {}_cE_\psi}$ to $\Gamma_r(q)^n$. In particular, it follows from the definition of c_r that $[c_{v_r^* {}_cE_\psi}] = c_r$. □

Fix $z \in \mathscr{X}$ an arbitrary point. Define a cocycle

$$\mu_{v_r^* {}_cE_\psi} : \Gamma^n \longrightarrow \mathbb{D}(\mathbb{X}_r, \mathbb{R}),\ (\gamma_0, \ldots, \gamma_{n-1}) \longmapsto \left(\bar{x} \longmapsto \int_{\Delta(\gamma_0 z, \ldots, \gamma_{n-1} z)} (x/p^r)^* {}_cE_\psi \right),$$

where $x \in \mathbb{Z}^n$ is a lift of $\bar{x} \in \mathbb{X}_r$, $z \in \mathscr{X}$ denotes a fixed arbitrary base point, and $\Delta(\gamma_0 z, \ldots, \gamma_{n-1} z)$ is defined as in the lemma above.

Proposition 4.10. *We have $[\mu_{v_r^* {}_cE_\psi}] = \mu_r$ when viewed as classes in $H^{n-1}(\Gamma, \mathbb{D}(\mathbb{X}_r, \mathbb{R}))$.*

Proof. First observe that $\mu_{v_r^* {}_cE_\psi}$ is a group cocycle. This follows from the fact that ${}_cE_\psi$ is closed and invariant under Γ. Now, the proposition follows from observing that $[\mu_{v_r^* {}_cE_\psi}]$ maps to $[c_{v_r^* {}_cE_\psi}]$ via the isomorphism given by Shapiro's lemma

$$H^{n-1}(\Gamma, \mathbb{D}(\mathbb{X}_r, \mathbb{R})) \xrightarrow{\sim} H^{n-1}(\Gamma_r, \mathbb{R})$$

described in (11), Lemma 4.9, and the definition of μ_r (see Definition 4.3). □

Consider the Γ-equivariant morphism given by taking the total mass of a distribution

$$\mathbb{D}(\mathbb{X}_1, \mathbb{R}) \longrightarrow \mathbb{R},\ \lambda \longmapsto \sum_{x \in \mathbb{X}_1} \lambda(x).$$

Corollary 4.11. *The corestriction map $H^{n-1}(\Gamma_1, \mathbb{R}) \to H^{n-1}(\Gamma, \mathbb{R})$ maps c_1 to 0. In particular, the morphism induced by taking the total mass of a measure*

$$H^{n-1}(\Gamma_1, \mathbb{D}(\mathbb{X}_1, \mathbb{R})) \longrightarrow H^{n-1}(\Gamma, \mathbb{R})$$

maps μ_1 to 0.

Proof. The corestriction map can be written as

$$H^{n-1}(\Gamma_1, \mathbb{R}) \xrightarrow{\sim} H^{n-1}(\Gamma, \mathbb{D}(\mathbb{X}_1, \mathbb{R})) \longrightarrow H^{n-1}(\Gamma, \mathbb{R}),$$

where the first map is given by the inverse of the map given by Shapiro's lemma, and the second one is the map induced by taking the total mass of a measure (see [Bro82, Chapter III, Section 9 (A)]). In view of this observation and of Proposition 4.10, it is enough to prove that the image of $[\mu_{v_1^* {}_cE_\psi}]$ by the second map is trivial. For that, observe that such image is represented by the cocycle

$$(\gamma_0, \ldots, \gamma_{n-1}) \longmapsto \int_{\Delta(\gamma_0 z, \ldots, \gamma_{n-1} z)} \sum_{\bar{x} \in \mathbb{X}_1} (x/p)^* {}_cE_\psi.$$

It follows from Proposition 3.10 that the sum of differential forms in the integral is equal to zero, giving the desired result. □

4.4. **Lifting to measures of total mass zero.** To construct rigid classes, it is useful to lift the class μ to a class with coefficients in measures of total mass zero. Let $\mathbb{D}_0 := \mathbb{D}_0(\mathbb{X}, \mathbb{Z}[1/m])$ be the sub-module of $\mathbb{D} := \mathbb{D}(\mathbb{X}, \mathbb{Z}[1/m])$ consisting of measures $\lambda \in \mathbb{D}$ such that $\lambda(\mathbb{X}) = 0$. Consider the short exact sequence

$$0 \longrightarrow \mathbb{D}_0 \longrightarrow \mathbb{D} \longrightarrow \mathbb{Z}[1/m] \longrightarrow 0. \tag{15}$$

Proposition 4.12. *The image of μ by the map $H^{n-1}(\Gamma, \mathbb{D}) \to H^{n-1}(\Gamma, \mathbb{Z}[1/m])$ is torsion.*

Proof. The result follows from Proposition 4.11. □

As we explained above, $w = \mathrm{diag}(1, -1, 1, \dots, 1) \in \mathrm{GL}_n(\mathbb{Z})$ acts on the cohomology groups $H^i(\Gamma, \mathbb{Z}[1/m])$, $H^i(\Gamma, \mathbb{D}_0)$, and $H^i(\Gamma, \mathbb{D})$. Moreover, (15) yields a long exact sequence

$$0 = H^{n-2}(\Gamma, \mathbb{Q})^- \longrightarrow H^{n-1}(\Gamma, \mathbb{D}_0)^-_{\mathbb{Q}} \longrightarrow H^{n-1}(\Gamma, \mathbb{D})^-_{\mathbb{Q}} \longrightarrow H^{n-1}(\Gamma, \mathbb{Q})^-,$$

where the subindex denotes taking the tensor product with $\mathbb{Q}$ over $\mathbb{Z}[1/m]$ and we used Lemma 4.5 for the vanishing of $H^{n-2}(\Gamma, \mathbb{Q})^-$. Thus, by Proposition 4.12, μ admits a unique lift to a class in $H^{n-1}(\Gamma, \mathbb{D}_0)^-_{\mathbb{Q}}$.

Definition 4.13. Let

$$\mu_0 \in H^{n-1}(\Gamma, \mathbb{D}_0(\mathbb{X}, \mathbb{Z}[1/m]))^-_{\mathbb{Q}} \tag{16}$$

be a a lift of $\mu \in H^{n-1}(\Gamma, \mathbb{D})_{\mathbb{Q}}$.

Remark 4.14. If a is any integer prime to p, we denote $[a]_*$ as the $\mathrm{GL}_n(\mathbb{Z})$-equivariant operator on $\mathbb{D}(\mathbb{X}_r)$ or $\mathbb{D}(\mathbb{X})$ given by pushforward of measures along the multiplication-by-a map $\mathbb{X}_r \to \mathbb{X}_r$ (resp. $\mathbb{X} \to \mathbb{X}$). Then Remark 2.10 implies that for any c and d coprime to p and each other

$$([c]_*^{-1} - c^n)_d \mu_r = ([d]_*^{-1} - d^n)_c \mu_r,$$

where the pre-subscripts, as before, denote the class associated to the corresponding smoothings. Then Proposition 4.6 implies the same for the inverse limit class μ. Note that the pullback $[a]^*$ on the cohomology of the torus induces $[a]_*^{-1}$ on the distributions over torsion specializations. From there, we deduce that, up to torsion,

$$([c]_*^{-1} - c^n)_d \mu_0 = ([d]_*^{-1} - d^n)_c \mu_0.$$

5. Drinfeld's symmetric domain and log-rigid classes

In this section, we introduce Drinfeld's p-adic symmetric domain $\mathscr{X}_p$. Then, we define a lift from measures on $\mathbb{X} = \mathbb{Z}_p^n - p\mathbb{Z}_p^n$ of total mass zero to log-rigid analytic functions on $\mathscr{X}_p$. This leads to construct a log-rigid class $J_{E,\mathcal{L}}$ as the image of the class $\mu_0 \in H^{n-1}(\Gamma, \mathbb{D}_0(\mathbb{X}, \mathbb{Z}[1/m]))_{\mathbb{Q}}$ of the previous section by such lift. We conclude by defining the evaluation of $J_{E,\mathcal{L}}$ at points $\tau \in \mathscr{X}_p$ attached to totally real fields where p is inert.

5.1. Drinfeld's domain and rigid functions.

Drinfeld's p-adic symmetric domain is defined as $\mathscr{X}_p := \mathbb{P}^{n-1}(\mathbb{C}_p) - \bigcup_{H \in \mathcal{H}} H$, where $\mathcal{H}$ is the set of all $\mathbb{Q}_p$-rational hyperplanes. It has the structure of a rigid analytic space, which we proceed to describe following [SS91].

For a given $H \in \mathcal{H}$, let ℓ_H be an equation of H such that its coefficients form a unimodular vector in $\mathbb{C}_p^n$. Also, if $z \in \mathbb{P}^{n-1}(\mathbb{C}_p)$, we will always assume $z = [(z_0, \dots, z_{n-1})]$ is represented by a vector with unimodular coordinates. For $m \geq 1$, define

$$\mathscr{X}_p^{\leq m} := \{z \in \mathbb{P}^{n-1}(\mathbb{C}_p) \mid \mathrm{ord}_p(\ell_H(z)) \leq m, \text{ for all } H \in \mathcal{H}\}.$$

The family $\{\mathscr{X}_p^{\leq m}\}_m$ forms an admissible covering of $\mathscr{X}_p$ by open affinoid subdomains.

The ring of rigid functions on $\mathscr{X}_p^{\leq m}$ can be described as follows. Let $\mathcal{H}_m$ be the set of equivalence classes of $\mathcal{H}$ modulo p^m. Also, fix $\bar{\mathcal{H}}_m$ a set of representatives for the equivalence classes in $\mathcal{H}_{m+1}$ containing the coordinate hyperplanes $H_i = \{z_i = 0\}$ for every $i = 0, \dots, n-1$. For $H, H' \in \mathcal{H}$, define the function $f_{H,H'}\colon \mathscr{X}_p \to \mathbb{C}_p$

$$f_{H,H'}(z) := \frac{\ell_H(z)}{\ell_{H'}(z)}.$$

Then, observe that we can describe

$$\mathscr{X}_p^{\leq m} = \{z \in \mathscr{X}_p \mid \mathrm{ord}_p(f_{H,H'}(z)) \geq -m \text{ for all } H, H' \in \bar{\mathcal{H}}_m\}.$$

Let A_m be the affinoid $\mathbb{Q}_p$-algebra obtained as the quotient of the free Tate algebra over $\mathbb{Q}_p$ in the indeterminates $\{T_{H,H'}\}_{H,H' \in \bar{\mathcal{H}}_m}$ modulo the closed ideal generated by

$$T_{H,H} - p^m, \text{ for } H \in \bar{\mathcal{H}}_m$$
$$T_{H,H'}T_{H',H''} - p^m T_{H,H''}, \text{ for } H, H', H'' \in \bar{\mathcal{H}}_m,$$
$$T_{H,H_j} - \sum_{i=0}^{r-1} \lambda_i T_{H_i,H_j}, \text{ if } \ell_H(z) = \sum_{i=0}^{n-1} \lambda_i z_i \text{ for } H \in \bar{\mathcal{H}}_m \text{ and } 0 \leq j \leq n-1.$$

The previous descriptions of $\mathscr{X}_p^{\leq m}$ and A_m lead to the following result.

Proposition 5.1. *Denote by $\mathcal{A}^{\leq m}$ the ring of rigid analytic functions on $\mathscr{X}_p^{\leq m}$. Then, we have an isomorphism of $\mathbb{Q}_p$-algebras*

$$A_m \xrightarrow{\sim} \mathcal{A}^{\leq m}, \ T_{H,H'} \longmapsto p^m f_{H,H'}.$$

In particular, it induces an isomorphism of rigid spaces $\mathscr{X}_p^{\leq m} \xrightarrow{\sim} \mathrm{Sp}(A_m)$.

Proof. See proof of Proposition 4 of [SS91]. □

In particular, $\mathcal{A}^{\leq m}$ is a Banach algebra with respect to the supremum norm.

Definition 5.2. The ring of rigid analytic functions on $\mathscr{X}_p$, denoted by $\mathcal{A}$, is the space of functions $f\colon \mathscr{X}_p \to \mathbb{C}_p$ such that for every m, their restriction to $\mathscr{X}_p^{\leq m}$ belongs to $\mathcal{A}^{\leq m}$.

We will also consider a larger space of functions on $\mathscr{X}_p$, called log-rigid analytic functions. Let $\log_p\colon \mathbb{C}_p^\times \to \mathbb{C}_p$ be the branch of the p-adic logarithm satisfying $\log_p(p) = 0$. A function $f\colon \mathscr{X}_p^{\leq m} \to \mathbb{C}_p$ is log-rigid analytic on $\mathscr{X}_p^{\leq m}$ if it can be written as

$$f = f_0 + \sum_{H,H' \in \mathcal{H}} c_{H,H'} \log_p(f_{H,H'}(z)),$$

where $f_0 \in \mathcal{A}^{\leq m}$ and $c_{H,H'} \in \mathbb{Q}_p$ are all but finitely many equal to 0. Denote the space of log-rigid analytic functions on $\mathscr{X}_p^{\leq m}$ by $\mathcal{A}_{\mathcal{L}}^{\leq m}$.

Definition 5.3. The space of log-rigid analytic functions on $\mathscr{X}_p$, denoted by $\mathcal{A}_{\mathcal{L}}$, is the space of functions $f\colon \mathscr{X}_p \to \mathbb{C}_p$ such that for every m, their restriction to $\mathscr{X}_p^{\leq m}$ belongs to $\mathcal{A}_{\mathcal{L}}^{\leq m}$.

The following lemma will be useful to study log-rigid functions in the next sections.

Lemma 5.4. *Let $m \geq 1$, and let $H, H' \in \mathcal{H}$ be hyperplanes with equations ℓ_H and $\ell_{H'}$ which are congruent modulo p^{m+1}. Then, the function*

$$f\colon \mathscr{X}_p^{\leq m} \longrightarrow \mathbb{C}_p, \ z \longmapsto \log_p\left(f_{H,H'}(z)\right)$$

is rigid analytic on $\mathscr{X}_p^{\leq m}$.

Proof. Observe that we can write

$$f(z) = \log_p \left(1 - \frac{\ell_{H'}(z) - \ell_H(z)}{\ell_{H'}(z)}\right).$$

Moreover, since $\ell_H \equiv \ell_{H'} \mod p^{m+1}$ and $z \in \mathscr{X}_p^{\leq m}$, we have

$$\mathrm{ord}_p \left(\frac{\ell_{H'}(z) - \ell_H(z)}{\ell_{H'}(z)}\right) \geq 1.$$

Therefore,

$$f(z) = \sum_{k\geq 1} \frac{1}{k} \left(\frac{\ell_{H'}(z) - \ell_H(z)}{\ell_{H'}(z)}\right)^k,$$

which is rigid analytic on $\mathscr{X}_p^{\leq m}$. □

Observe that matrix multiplication induces a right action of $\mathrm{SL}_n(\mathbb{Q}_p)$ on $\mathscr{X}_p$ given as follows. For $g \in \mathrm{SL}_n(\mathbb{Q}_p)$ and $z \in \mathscr{X}_p$ represented by a vector in $\mathbb{C}_p^n$, that we also denote by z, we have

$$(z, g) := [g^t z],$$

where $g^t \in \mathrm{SL}_n(\mathbb{Q}_p)$ denotes the transpose of g. This induces a left action of $\mathrm{SL}_n(\mathbb{Q}_p)$ on the space of $\mathbb{C}_p$-valued functions on $\mathscr{X}_p$. If $g \in \mathrm{SL}_n(\mathbb{Q}_p)$, f is a function on $\mathscr{X}_p$, and $z \in \mathscr{X}_p$

$$(g \cdot f)(z) := f(g^t z).$$

This action preserves the subspaces $\mathcal{A}$ and $\mathcal{A}_{\mathcal{L}}$.

5.2. **Lifts from measures to functions on $\mathscr{X}_p$.** Recall that $\mathscr{X}_p$ consists of the points in $\mathbb{P}^{n-1}(\mathbb{C}_p)$ that do not belong to a $\mathbb{Q}_p$-rational hyperplane. On the other hand, a point in $\mathbb{X} = \mathbb{Z}_p^n - p\mathbb{Z}_p^n$ gives the equation of a $\mathbb{Q}_p$-rational hyperplane. This suggests considering the two-variable function

$$\left(\mathbb{C}_p^n - \bigcup_{H\in\mathcal{H}} H\right) \times \mathbb{X} \longrightarrow \mathbb{C}_p,\ (z, x) \longmapsto \log_p(z^t \cdot x),$$

Integration with respect to the variable $x \in \mathbb{X}$ will induce a map from total mass zero measures on $\mathbb{X}$ to functions on $\mathscr{X}_p$.

Lemma 5.5. *Let $\lambda \in \mathbb{D}_0(\mathbb{X}, \mathbb{Z}_p)$. The function $F\colon \mathscr{X}_p \longrightarrow \mathbb{C}_p$ given by*

$$z \longmapsto F(z) := \int_{\mathbb{X}} \log_p(z^t \cdot x) d\lambda,$$

where z in the right hand side denotes an arbitrary representative in $\mathbb{C}_p^n$ of $z \in \mathscr{X}_p$, is well-defined and belongs to $\mathcal{A}_{\mathcal{L}}$.

Proof. For every $r \geq 1$, fix V_r a set of representatives in $\mathbb{Z}^n$ of $\mathbb{X}_r = (\mathbb{Z}/p^r\mathbb{Z})^n - (p\mathbb{Z}/p^r\mathbb{Z})^n$ and define

$$f_r\colon \mathscr{X}_p \longrightarrow \mathbb{C}_p,\ z \longmapsto \sum_{v\in V_r} \lambda(U_{v/p^r}) \log_p(z^t \cdot v),$$

where $U_{v/p^r} \subset \mathbb{X}$ is as in (13). Observe that since $\lambda(\mathbb{X}) = 0$, $f_r(z)$ is independent of the choice of representative of z in $\mathbb{C}_p^n$, showing that f_r is a well-defined function. For the rest of the proof we will assume that the representative of z (also denoted z) is chosen so that its coordinates are unimodular. We follow the next steps:

- F is a well-defined function on $\mathscr{X}_p$. Indeed, for $z \in \mathbb{C}_p^n - \bigcup_{H \in \mathcal{H}} H$, the function $x \in \mathbb{X} \mapsto \log_p(z^t \cdot x)$ is continuous on the compact set $\mathbb{X}$. Thus, the integral defining $F(z)$ converges and we have pointwise convergence
$$F(z) = \lim_{r \to +\infty} f_r(z).$$
- The sequence $(f_{r|\mathscr{X}_p^{\leq m}})$ converges to $F_{|\mathscr{X}_p^{\leq m}}$ with respect to the sup norm for $m \geq 1$. To simplify the notation, denote by (f_r) and F the restrictions of these functions to $\mathscr{X}_p^{\leq m}$. To prove that $(f_r)_r$ converges to F with respect to the sup norm it is enough to see that $(f_r)_r$ is Cauchy with respect to this norm. Observe that, if we let $\pi\colon V_{r+1} \twoheadrightarrow V_r$ be the lift of the reduction modulo p^r map $\mathbb{X}_{r+1} \twoheadrightarrow \mathbb{X}_r$ and use that λ is a measure, we have
$$\begin{aligned} f_{r+1}(z) - f_r(z) &= \sum_{v \in V_{r+1}} \lambda(U_{v/p^{r+1}}) \log_p \left(\frac{z^t \cdot v}{z^t \cdot \pi(v)} \right) \\ &= \sum_{v \in V_{r+1}} \lambda(U_{v/p^{r+1}}) \log_p \left(1 + \frac{z^t \cdot (v - \pi(v))}{z^t \cdot \pi(v)} \right). \end{aligned}$$
Since $v \equiv \pi(v) \mod p^r$, we deduce that for every $z \in \mathscr{X}_p^{\leq m}$
$$\mathrm{ord}_p \left(\frac{z^t \cdot (v - \pi(v))}{z^t \cdot \pi(v)} \right) \geq r - m,$$
Thus, if $r > m$, we can use the power series expansion of $\log(1+x)$ to deduce that
$$\mathrm{ord}_p(f_{r+1}(z) - f_r(z)) \geq r - m \text{ for all } z \in \mathscr{X}_p^{\leq m}$$
It follows from there that $(f_r)_r$ is Cauchy.
- $F \in \mathcal{A}_{\mathcal{L}}$. Let $m \geq 1$ and denote by $(f_r)_r$ and F the restrictions of these functions to $\mathscr{X}_p^{\leq m}$. It is enough to see that F belongs to $\mathcal{A}_{\mathcal{L}}^{\leq m}$. With this aim, write
$$F = \left(\lim_{r \to +\infty} (f_r - f_{m+1}) \right) + f_{m+1}.$$
We claim that $\lim_{r \to +\infty}(f_r - f_{m+1})$ is a rigid analytic function. Indeed, we can write
$$f_r(z) - f_{m+1}(z) = \sum_{v \in V_{r+1}} \lambda(U_{v/p^{r+1}}) \log_p \left(\frac{z^t \cdot v}{z^t \cdot \pi^{r-(m+1)}(v)} \right).$$
Since $v \equiv \pi^{r-(m+1)}(v) \mod p^{m+1}$, it follows from Lemma 5.4, that $f_r - f_{m+1}$ is rigid analytic on $\mathscr{X}_p^{\leq m}$. Then, since the sequence $(f_r - f_{m+1})_r$ converges with respect to the sup norm by the previous point of this proof, and $\mathcal{A}^{\leq m}$ is complete with respect to this norm, we deduce the desired claim.

  On the other hand, since λ has total mass zero, we have that $f_{m+1} \in A_{\mathcal{L}}^{\leq m}$, as it can be written as a linear combination of $\log_p(f_{H,H'}(z))$ for $\mathbb{Q}_p$-rational hyperplanes $H, H' \in \mathcal{H}$. Hence, we deduce that $F \in \mathcal{A}_{\mathcal{L}}^{\leq m}$ and we are done.

$\square$

In view of the previous lemma, we can define a lift from measures of total mass zero to log-rigid analytic functions on $\mathscr{X}_p$.

Definition 5.6. Let ST be the morphism given by

$$\mathrm{ST}\colon \mathbb{D}_0(\mathbb{X},\mathbb{Z}[1/m]) \longrightarrow \mathcal{A}_{\mathcal{L}},\ \lambda \longmapsto \left(z \longmapsto \int_{\mathbb{X}} \log_p(z^t\cdot x)d\lambda\right).$$

The morphism ST is Γ-equivariant. Therefore, it induces a map in cohomology

$$\mathrm{ST}\colon H^{n-1}(\Gamma,\mathbb{D}_0(\mathbb{X},\mathbb{Z}[1/m])) \longrightarrow H^{n-1}(\Gamma,\mathcal{A}_{\mathcal{L}}).$$

Using this map, we obtain our desired log-rigid analytic class.

Definition 5.7. Let $\mu_0 \in H^{n-1}(\Gamma,\mathbb{D}_0(\mathbb{X},\mathbb{Z}[1/m]))_{\mathbb{Q}}$ be as in (16). Define

$$J_{E,\mathcal{L}} := \mathrm{ST}(\mu_0) \in H^{n-1}(\Gamma,\mathcal{A}_{\mathcal{L}})_{\mathbb{Q}}.$$

As with the cocycles μ_0, we have the following independence-of-c result, where we here, as before, denote dependence on the smoothing with a pre-subscript.

Proposition 5.8. *If c, d are coprime to each other and also to p, we have*

$$(1-d^n)_c J_{E,\mathcal{L}} = (1-c^n)_d J_{E,\mathcal{L}}.$$

Proof. For any prime-to-p scalar a, we have $\mathrm{ST}\circ[a]_* = \mathrm{ST}$, as if $\lambda \in \mathbb{D}_0(\mathbb{X},\mathbb{Z}[1/m])$

$$\int_{\mathbb{X}} \log_p(z^t\cdot x)\,d([a]_*\lambda) = \int_{\mathbb{X}} \log_p a + \log_p(z^t\cdot x)\,d\lambda = \int_{\mathbb{X}} \log_p(z^t\cdot x)\,d\lambda$$

with the last equality by $\lambda(\mathbb{X}) = 0$. Then the result follows by passing to group cohomology for Γ and applying Remark 4.14. □

In particular, $(1-c^n)^{-1}{}_c J_{E,\mathcal{L}}$ is independent of c, though this introduces denominators.

Remark 5.9. Suppose $n=2$ and consider $\mathcal{J}_{\mathrm{DR}} \in H^1(\mathrm{SL}_2(\mathbb{Z}),\mathcal{A}^\times)^-$ a lift of (the restriction to $\mathrm{SL}_2(\mathbb{Z})$ of) $J_{\mathrm{DR}} \in H^1(\mathrm{SL}_2(\mathbb{Z}[1/p]),\mathcal{A}^\times/\mathbb{C}_p^\times)^-$ constructed in [DPV24]. By comparing the constructions of J_{DR} and $J_{E,\mathcal{L}}$, we deduce $J_{E,\mathcal{L}} = \log_p(\mathcal{J}_{\mathrm{DR}})$.

5.3. **Evaluation at totally real fields where p is inert.** Let F be a totally real field of degree n where p is inert and denote by $\sigma_1,\dots,\sigma_n$ the collection of embeddings of F into $\mathbb{R}$. Let $\mathfrak{a}$ be an integral ideal of F of norm coprime to pc. Fix $\{\tau_1,\dots,\tau_n\}$ an oriented $\mathbb{Z}$-basis of $\mathfrak{a}^{-1}$, in the sense that the square matrix $(\sigma_i(\tau_j))_{i,j}$ has positive determinant, and let $\tau \in F^n$ be the column vector whose ith entry is equal to τ_i. The vector τ induces an isomorphism of $\mathbb{Q}$-vector spaces

$$\mathbb{Q}^n \xrightarrow{\sim} F,\ x \longmapsto \tau^t\cdot x.$$

The action of multiplication by $F^\times$ on F, which is $\mathbb{Q}$-linear, gives an embedding

$$F \longhookrightarrow \mathrm{M}_n(\mathbb{Q}),\ \alpha \longmapsto A_\alpha \tag{17}$$

determined by the following property: for $\alpha \in F$ and $x \in \mathbb{Q}^n$, $\alpha(\tau^t\cdot x) = \tau^t\cdot(A_\alpha x)$.

Lemma 5.10. *The element $\tau \in \mathbb{P}^{n-1}(\mathbb{C}_p)$ belongs to $\mathscr{X}_p$ and is fixed by $F^1 \hookrightarrow \mathrm{SL}_n(\mathbb{Q})$.*

Proof. The coordinates of τ give a $\mathbb{Q}$-basis of F. Since p is inert in F, the coordinates of τ also form a $\mathbb{Q}_p$-basis of the completion of F at p. In particular, they are independent over $\mathbb{Q}_p$. In other words, $\tau \in \mathscr{X}_p$. Finally, for every $\alpha \in F$ we have $A_\alpha^t\tau = \alpha\tau$ by the property stated below (20). In particular, $\tau \in \mathscr{X}_p$ is fixed by the action of $F^1 \hookrightarrow \mathrm{SL}_n(\mathbb{Q})$. □

Let U_F be the subgroup of totally positive units in $\mathcal{O}_F^\times$. We view U_F as a subgroup of Γ. Consider the following morphism in cohomology induced by evaluation at τ

$$H^{n-1}(\Gamma, \mathcal{A}_{\mathcal{L}}) \xrightarrow{\mathrm{ev}_\tau} H^{n-1}(U_F, \mathbb{C}_p).$$

By Dirichlet's unit theorem, $U_F \simeq \mathbb{Z}^{n-1}$. Therefore, $H_{n-1}(U_F, \mathbb{Z}) \simeq \mathbb{Z}$, and we can fix a generator of this group $c_{U_F} \in H_{n-1}(U_F, \mathbb{Z})$.

Definition 5.11. Consider the same notation as above, and let $J \in H^{n-1}(\Gamma, \mathcal{A}_{\mathcal{L}})_{\mathbb{Q}}$. Define the evaluation of J at $[\tau] \in \mathscr{X}_p$ by the cap product

$$J[\tau] := c_{U_F} \frown \mathrm{ev}_\tau(J) \in \mathbb{C}_p.$$

Observe that, since $J_{E,\mathcal{L}} = \mathrm{ST}(\mu_0)$, it follows from the description of the map ST that $J_{E,\mathcal{L}}[\tau] \in F_p$. We also note that this definition depends, up to a sign, of the choice of generator $c_{U_F} \in H_{n-1}(U_F, \mathbb{Z})$. In the next section, we will make a precise choice of generator when comparing the local trace of these values to the local trace of p-adic logarithms of Gross–Stark units.

6. Traces of values of the log-rigid class and the Gross–Stark Conjecture

Let F be a totally real field where p is inert, let $\mathfrak{a}$ be an integral ideal of F coprime to pc, and fix $\tau \in F^n$ a vector whose entries give an oriented $\mathbb{Z}$-basis of $\mathfrak{a}^{-1}$, which yields a point $\tau \in X_p$. Recall the log-rigid analytic class $J_{E,\mathcal{L}}$ constructed in the previous section and the value $J_{E,\mathcal{L}}[\tau] \in F_p$. In this section, we prove

$$\mathrm{Tr}_{F_p/\mathbb{Q}_p} J_{E,\mathcal{L}}[\tau] = -L_p'(1_{[\mathfrak{a}],p}, 0),$$

where $L_p(1_{[\mathfrak{a}],p}, s)$ denotes a p-adic partial zeta function attached to the class of $\mathfrak{a}$ in the narrow Hilbert class group of F. From this expression and the rank 1 Gross–Stark conjecture, we obtain the equality

$$\mathrm{Tr}_{F_p/\mathbb{Q}_p} J_{E,\mathcal{L}}[\tau] = \mathrm{Tr}_{F_p/\mathbb{Q}_p} \log_p(u^{\sigma_{\mathfrak{a}}})$$

for $u^{\sigma_{\mathfrak{a}}}$ a Gross-Stark unit in the narrow Hilbert class field of F attached to the class of $\mathfrak{a}$.

6.1. p-adic L-functions and Gross–Stark conjecture.

We state the the Gross–Stark conjecture in a simple setting. For more details, we refer the reader to [Das08, Section 2]. We begin by introducing the following notation. For an integral ideal $\mathfrak{f}$ of F, denote by $G_{\mathfrak{f}}$ the narrow ray class group modulo $\mathfrak{f}$. It is obtained by taking the quotient of the set of integral ideals in F which are prime to $\mathfrak{f}$ by the relation

$$\mathfrak{b} \sim_{\mathfrak{f}} \mathfrak{c} \text{ if and only if } \mathfrak{b}\mathfrak{c}^{-1} = (\lambda) \text{ for } \lambda \in 1 + \mathfrak{f}\mathfrak{c}^{-1} \text{ totally positive.}$$

Then, if ε is a $\bar{\mathbb{Q}}$-valued function on $G_{\mathfrak{f}}$, we let

$$L(\varepsilon, s) := \sum_{(\mathfrak{b},\mathfrak{f})=1} \varepsilon(\mathfrak{b}) \mathrm{N}\mathfrak{b}^{-s},$$

where the sum is over integral ideals which are coprime to $\mathfrak{f}$. This sum converges for $s \in \mathbb{C}$ such that $\mathrm{Re}(s) > 1$ and it can be extended via analytic continuation to a meromorphic function at $\mathbb{C}$ with at most a pole at $s = 1$, that we will still denote by $L(\varepsilon, s)$. Recall that c

is a positive integer prime to p and denote by ε_c the function on $G_{\mathfrak{f}}$ given by $\varepsilon_c(\mathfrak{b}) = \varepsilon((c)\mathfrak{b})$. For $k \in \mathbb{Z}_{\geq 1}$, consider

$$\Delta_c(\varepsilon, 1-k) := L(\varepsilon, 1-k) - c^{nk} L(\varepsilon_c, 1-k).$$

It is result of Klingen and Siegel that $\Delta_c(\varepsilon, 1-k) \in \mathbb{Q}(\varepsilon)$, where $\mathbb{Q}(\varepsilon)$ denotes the field generated by the values of ε. Deligne–Ribet and Cassou–Noguès refined this statement by studying the integrality properties of these values. Their study results in the existence of p-adic analytic functions interpolating these values, which we proceed to outline for the case of partial zeta functions.

Let $G := \varprojlim_{r\geq 1} G_{p^r}$, where the limit is taken with respect to the natural projection maps $G_{p^{r+1}} \to G_{p^r}$, let $H_{\mathfrak{a}}$ be the open subset of G consisting of the pre-image of $\mathfrak{a}$ via the natural map $G \to G_1$, and denote by $1_{[\mathfrak{a}],p}\colon G \to \mathbb{Z}$ the characteristic function of $H_{\mathfrak{a}}$. If $\varepsilon\colon G \to \mathbb{Z}$ is locally constant, it factors through G_{p^r} for some $r \geq 1$. We then define $L(\varepsilon, s)$ by viewing ε as a function on G_{p^r}, which is independent of the choice of r.

Theorem 6.1. *For $\varepsilon\colon H_{\mathfrak{a}} \to \mathbb{Z}$ locally constant, consider the product $\varepsilon 1_{[\mathfrak{a}],p}$ and view it as a locally constant function on G.*

(1) If $k \geq 1$, we have $\Delta_c(\varepsilon 1_{[\mathfrak{a}],p}, 1-k) \in \mathbb{Z}[1/c]$.
(2) The distribution $\mu_{\mathfrak{a}}\colon \varepsilon \mapsto \Delta_c(\varepsilon 1_{[\mathfrak{a}],p}, 0)$ defines a measure on $H_{\mathfrak{a}}$.
(3) The function

$$L_p(1_{[\mathfrak{a}],p}, \cdot)\colon \mathbb{Z}_p \longrightarrow \mathbb{Z}_p,\ s \longmapsto \int_{H_{\mathfrak{a}}} \langle \mathrm{N}\mathfrak{b}\rangle^{-s} d\mu_{\mathfrak{a}}(\mathfrak{b})$$

is analytic and is characterized by the following interpolation property: for every integer $k \geq 1$ such that $k \equiv 1 \mod [F(\mu_{2p}) : F]$,

$$L_p(1_{[\mathfrak{a}],p}, 1-k) = \Delta_c(1_{[\mathfrak{a}],p}, 1-k). \tag{18}$$

Proof. See Theorem 0.5 of [DR80]. □

Observe that $L(1_{[\mathfrak{a}],p}, s)$ is a partial zeta function with the Euler factor corresponding to p removed. This implies that $\Delta_c(1_{[\mathfrak{a}],p}, 0) = 0$ and, by (18), $L_p(1_{[\mathfrak{a}],p}, 0) = 0$ as well. The Gross–Stark conjecture gives an arithmetic interpretation for the value of the derivative $L_p'(1_{[\mathfrak{a}],p}, 0)$ with respect to s at $s = 0$. For that, let H be the narrow Hilbert class field of F and consider the following subgroup of p-units in H

$$\mathcal{O}_H[1/p]_-^\times := \{x \in H^\times \mid |x|_{\mathfrak{q}} = 1\ \forall \mathfrak{q} \nmid p\},$$

where $\mathfrak{q}$ runs over all archimedean and nonarchimedean places of H not dividing p. Fix $\mathfrak{p}$ a prime of H dividing p.

Proposition 6.2. *There exists a unique element $u \in \mathcal{O}_H[1/p]_-^\times \otimes \mathbb{Q}$ satisfying*

$$\mathrm{ord}_{\mathfrak{p}}(u^{\sigma_{\mathfrak{a}}}) = \Delta_c(1_{[\mathfrak{a}]}, 0) \text{ for all } \mathfrak{a} \text{ coprime to } p,$$

where $1_{[\mathfrak{a}]}$ denotes the characteristic function of $[\mathfrak{a}]$ on G_1 and, here and from now on, $\sigma_{\mathfrak{a}} \in \mathrm{Gal}\,(H/F)$ denotes the Frobenius element associated to $\mathfrak{a}$.

Remark 6.3. If every prime factor of c is greater than $n+1$, the Brumer–Stark conjecture, proven in [DK23] and [Das+23], implies that in fact $u \in \mathcal{O}_H[1/p]^\times_-$. Indeed, the quantities $\Delta_c(\varepsilon, 1-k)$ can be written as a linear combination of values of (smoothed) partial zeta functions $\zeta_{S,T}(\sigma, 1-k)$ considered in [Das08] for T running over subsets of the set of prime ideals of F dividing $c\mathcal{O}_F$. Under the condition on c given above, each of these subsets satisfies the assumptions to apply the proof of Brumer–Stark, see [Das+23, Section 1.1].

Since $p\mathcal{O}_F$ splits completely on H, we have $H \subset H_{\mathfrak{p}} = F_p$.

Theorem 6.4 (Gross–Stark conjecture)**.** *Let u be as in Proposition 6.2. We have*

$$L_p'(1_{[\mathfrak{a}],p}, 0) = -\log_p(\mathrm{N}_{F_p/\mathbb{Q}_p} u^{\sigma_{\mathfrak{a}}}) \text{ for all } \mathfrak{a} \text{ coprime to } p.$$

Proof. See [DDP11] and [Ven15]. □

6.2. **Periods of the Eisenstein class along tori attached to totally real fields.** We use the differential forms representing the Eisenstein class of Section 3 to prove that pullbacks of the Eisenstein class by torsion sections encode special values of zeta functions of totally real fields. A general version of the result was proven in [BCG20, Section 12.6] using an adelic framework, and we specialize their results and outline the proof below for the cases that will be relevant for us. Our calculations are similar to those in Section 4.2 of [BCG23].

Recall that F is a totally real field of degree n where p is inert, $\mathfrak{a}$ is an integral ideal of F prime to pc, and $\tau \in F^n$ is a column vector whose entries give a positively oriented $\mathbb{Z}$-basis of $\mathfrak{a}^{-1}$. As we saw in the previous section, τ induces a $\mathbb{Q}$-linear isomorphism

$$\beta\colon \mathbb{Q}^n \xrightarrow{\sim} F,\ x \longmapsto \tau^t \cdot x. \tag{19}$$

The action of multiplication by $F^\times$ on F, which is $\mathbb{Q}$-linear, gives an embedding

$$F \hookrightarrow \mathrm{M}_n(\mathbb{Q}),\ \alpha \longmapsto A_\alpha \tag{20}$$

determined by the following property: for all $\alpha \in F$ and $x \in \mathbb{Q}^n$, $\alpha(\tau^t \cdot x) = \tau^t \cdot (A_\alpha x)$. Let $(F \otimes \mathbb{R})^1_+$ be the subset of totally positive elements of norm 1. The embedding (20) induces an oriented map (see Section 12.4 of [BCG20] for more details on the orientation)

$$i_\tau\colon (F \otimes \mathbb{R})^1_+ \longrightarrow \mathscr{X}.$$

Denote by U_F the subgroup of totally positive units in $\mathcal{O}_F^\times$. Since U_F has rank $n-1$ by Dirichlet's unit theorem, it follows that

$$X(F) := U_F \backslash (F \otimes \mathbb{R})^1_+ \tag{21}$$

is a compact oriented manifold of dimension $n-1$.

We now introduce a linear combination of pullbacks of the Eisenstein class that we will integrate along $X(F)$. For $r \geq 1$, let

$$\chi\colon \left(\mathfrak{a}^{-1} - p\mathfrak{a}^{-1}\right)/p^r\mathfrak{a}^{-1} \longrightarrow \bar{\mathbb{Q}}$$

be an $\mathcal{O}_F^\times$-invariant function, where here and from now on, $\left(\mathfrak{a}^{-1} - p\mathfrak{a}^{-1}\right)/p^r\mathfrak{a}^{-1}$ denotes the set $\mathfrak{a}^{-1} - p\mathfrak{a}^{-1}$ modulo the translation action by $p^r\mathfrak{a}^{-1}$. Recall $\mathbb{X}_r := (\mathbb{Z}/p^r\mathbb{Z})^n - (p\mathbb{Z}/p^r\mathbb{Z})^n$

and observe that dot product with τ induces a bijection

$$\mathbb{X}_r \xrightarrow{\sim} \left(\mathfrak{a}^{-1} - p\mathfrak{a}^{-1}\right)/p^r\mathfrak{a}^{-1}. \tag{22}$$

We will sometimes view χ as a function on $\mathfrak{a}^{-1}$ which is 0 on $p\mathfrak{a}^{-1}$, and the rest of their values are determined by the value of χ on the residue classes modulo $p^r\mathfrak{a}^{-1}$.

Definition 6.5. Consider the same notation as above. Define

$$E_{\tau,\chi} := \sum_{\bar{x}\in\mathbb{X}_r} \chi(c\tau^t \cdot x)\,(x/p^r)^*\,{}_cE_\psi \in \Omega^{n-1}(\mathscr{X}), \tag{23}$$

For $s \in \mathbb{C}$, we define $E_{\tau,\chi}(s)$ as above but replacing ${}_cE_\psi$ by ${}_cE_\psi(s) = E_\psi(c^{-1}\mathbb{Z}^n, s) - c^nE_\psi(\mathbb{Z}^n, s)$ in the definition.

Lemma 6.6. *The differential form $E_{\tau,\chi}$ on $\mathscr{X}$ is invariant under $U_F \subset \Gamma$, where the inclusion of U_F in Γ is induced by* (20)*.*

Proof. For $\gamma \in \Gamma$, note that we have $\gamma^*v^*{}_cE_\psi = (\gamma v)^*{}_cE_\psi$. Then, if $\gamma \in U_F \subset \Gamma$

$$\begin{aligned}
\gamma^*E_{\tau,\chi} &:= \sum_{\bar{x}\in\mathbb{X}_r} \chi(c\tau^t \cdot x)(\gamma x/p^r)^*{}_cE_\psi \\
&= \sum_{\bar{x}\in\mathbb{X}_r} \chi(c\tau^t \cdot \gamma^{-1}x)(x/p^r)^*{}_cE_\psi \\
&= \sum_{\bar{x}\in\mathbb{X}_r} \chi(\varepsilon c\tau^t \cdot x)(x/p^r)^*{}_cE_\psi \\
&= \sum_{\bar{x}\in\mathbb{X}_r} \chi(c\tau^t \cdot x)(x/p^r)^*{}_cE_\psi,
\end{aligned}$$

where we used that $\tau^t\gamma^{-1} = \varepsilon\tau^t$, for $\varepsilon \in U_F$ the preimage of γ^{-1} by (20) and that χ is U_F-invariant. □

Thus, $i_\tau^*E_{\tau,\chi}$ defines a closed form on $X(F)$ and we can consider

$$\int_{X(F)} i_\tau^*E_{\tau,\chi}.$$

We will express this integral in terms of L-values. Observe that we have a bijection

$$\left(\left(\mathfrak{a}^{-1} - p\mathfrak{a}^{-1}\right)/p^r\mathfrak{a}^{-1}\right)/U_F \xrightarrow{\sim} \{\mathfrak{b} \in G_{p^r} \mid \mathfrak{b} \sim_1 \mathfrak{a}\} \longrightarrow G_{p^r},\ [\lambda] \longmapsto [\mathfrak{a}(\lambda)], \tag{24}$$

where $\lambda \in \mathfrak{a}^{-1}$ is a totally positive element in $[\lambda]$. We can use this bijection to consider

$$\chi \cdot 1_{[\mathfrak{a}],p} \colon G_{p^r} \longrightarrow \bar{\mathbb{Q}},$$

where $1_{[\mathfrak{a}],p}$ denotes characteristic function of the preimage of $[\mathfrak{a}] \in G_1$ via the projection $G_{p^r} \to G_1$ and χ is viewed as a function on the preimage of $[\mathfrak{a}]$ in G_{p^r} via the bijection above.

Lemma 6.7. *We have*

$$L(\chi 1_{[\mathfrak{a}],p}, 0) = \lim_{s\to 0} \frac{1}{2^n} \sum_{\alpha\in U_F\backslash\mathfrak{a}^{-1}} \frac{\chi(\alpha)\mathrm{sign}(\mathrm{N}\alpha)}{|\mathrm{N}\alpha|^s},$$

where on the right hand side, $\lim_{s\to 0}$ *denotes evaluation at $s = 0$ of the analytic continuation.*

Proof. The result can be deduced from equation (7.15) of [Cha07], which is originally due to Siegel ([Sie79]). □

Theorem 6.8. *Consider the same notation as above. Then,*

$$\int_{X(F)} i_\tau^* E_{\tau,\chi} = \Delta_c\left(\chi 1_{[\mathfrak{a}],p}, 0\right).$$

Proof. For $s \in \mathbb{C}$ such that $\mathrm{Re}(s) \gg 0$, we have

$$\int_{X(F)} i_\tau^* E_{r,\chi}(s) = \int_{X(F)} i_\tau^* \left(\sum_{\bar{v}\in\mathbb{X}_r} \chi(c\tau^t \cdot v)\,(v/p^r)^*\,{}_cE_\psi(s)\right)$$
$$= \int_{X(F)} i_\tau^* \left(\sum_{\bar{v}\in\mathbb{X}_r} \chi(c\tau^t\cdot v)\left(\sum_{\lambda\in v/p^r+c^{-1}\mathbb{Z}^n} \eta(\lambda,s) - c^n \sum_{\lambda\in v/p^r+\mathbb{Z}^n}\eta(\lambda,s)\right)\right),$$

where we recall that $\eta(\lambda, s)$ was introduced in Section 3.3. Using that $\eta(v/p^r, s) = p^s\eta(v, s)$, and keeping in mind that we will later be interested in evaluating the analytic continuation of the expression above at $s = 0$, it is enough to compute

$$\int_{X(F)} i_\tau^* \left(\sum_{\bar{v}\in\mathbb{X}_r} \chi(c\tau^t\cdot v)\left(\sum_{\lambda\in v+c^{-1}p^r\mathbb{Z}^n} \eta(\lambda,s) - c^n \sum_{\lambda\in v+p^r\mathbb{Z}^n}\eta(\lambda,s)\right)\right)$$
$$= \int_{X(F)} i_\tau^* \left(\sum_{\bar{v}\in\mathbb{X}_r} \chi(c\tau^t\cdot v)\left(\sum_{x\in\beta(v)+c^{-1}p^r\mathfrak{a}^{-1}} \eta(\beta^{-1}x,s) - c^n \sum_{x\in\beta(v)+p^r\mathfrak{a}^{-1}}\eta(\beta^{-1}x,s)\right)\right)$$
$$= \int_{X(F)} i_\tau^* \left(\sum_{x\in c^{-1}\mathfrak{a}^{-1}} \chi(cx)\eta(\beta^{-1}x,s) - c^n \sum_{x\in\mathfrak{a}^{-1}}\chi(cx)\eta(\beta^{-1}x,s)\right).$$

We can compute the inner sums by first taking representatives of $U_F\backslash c^{-1}\mathfrak{a}^{-1}$ and $U_F\backslash\mathfrak{a}^{-1}$, that we denote by x, and then running over all elements in U_F, denoted by u. Hence, we obtain that the previous expressions can be written as

$$\int_{X(F)} i_\tau^* \left(\sum_{U_F\backslash c^{-1}\mathfrak{a}^{-1}}\sum_{U_F} \chi(cux)\eta(\beta^{-1}ux,s) - c^n \sum_{U_F\backslash\mathfrak{a}^{-1}}\sum_{U_F}\chi(cux)\eta(\beta^{-1}ux,s)\right)$$
$$= \sum_{U_F\backslash c^{-1}\mathfrak{a}^{-1}} \chi(cx)\int_{(F\otimes\mathbb{R})^1_+} i_\tau^*\eta(\beta^{-1}x,s) - c^n \sum_{U_F\backslash\mathfrak{a}^{-1}} \chi(cx)\int_{(F\otimes\mathbb{R})^1_+} i_\tau^*\eta(\beta^{-1}x,s).$$

From [BCG20, Section 12.8], we have that for $x \in F$

$$\int_{(F\otimes\mathbb{R})^1_+} i_\tau^*\eta(\beta^{-1}x,s) = \pi^{-n/2}2^{s/2-n}\Gamma\left(\frac{s}{2n}+\frac{1}{2}\right)^n \frac{\mathrm{sign}(\mathrm{N}(x))}{|\mathrm{N}(x)|^s}.$$

Hence, we deduce

$$\int_{X(F)} i_\tau^* E_{r,\chi} = \frac{1}{2^n}\lim_{s\to 0} \sum_{x\in U_F\backslash\mathfrak{a}^{-1}} \chi(x)\frac{\mathrm{sign}(\mathrm{N}(x))}{|\mathrm{N}(x)|^s} - c^n \sum_{x\in U_F\backslash\mathfrak{a}^{-1}} \chi(cx)\frac{\mathrm{sign}(\mathrm{N}(x))}{|\mathrm{N}(x)|^s}.$$

Finally, the desired equality follows from Lemma 6.7. □

6.3. **The class μ and p-adic L-functions.** We state the relation between the class μ constructed in Section 4 and the p-adic L-function $L_p(1_{[\mathfrak{a}],p}, s)$ introduced above. From there, we relate $\mathrm{Tr}_{F_p/\mathbb{Q}_p} J_{E,\mathcal{L}}[\tau]$ to traces of p-adic logarithms of Gross–Stark units.

Denote by $\mathfrak{a}_p$ the completion of $\mathfrak{a}$ at p. For $\chi\colon \mathfrak{a}_p^{-1} - p\mathfrak{a}_p^{-1} \to \bar{\mathbb{Q}}_p$ a continuous function that is $\mathcal{O}_F^\times$-equivariant, define the map

$$\varphi_\chi\colon \mathbb{D}(\mathbb{X}, \mathbb{Z}[1/m]) \longrightarrow \bar{\mathbb{Q}}_p,\ \lambda \longmapsto \int_{\mathbb{X}} \chi(c\tau^t \cdot x) d\lambda.$$

Since φ_χ is U_F-equivariant, it induces a map in cohomology

$$\varphi_\chi\colon H^{n-1}(\Gamma, \mathbb{D}(\mathbb{X}, \mathbb{Z}[1/m])) \longrightarrow H^{n-1}(U_F, \bar{\mathbb{Q}}_p).$$

Fix the generator $c_{U_F} \in H_{n-1}(U_F, \mathbb{Z}) \simeq H_{n-1}(X(F), \mathbb{Z}) \simeq \mathbb{Z}$ corresponding to the positive orientation of $X(F)$ in (21). We can then consider the cap product

$$c_{U_F} \frown \varphi_\chi(\mu) \in \bar{\mathbb{Q}}_p.$$

To make the notation more transparent, we will write

$$c_{U_F} \frown \varphi_\chi(\mu) = \int_{\mathbb{X}} \chi(c\tau^t \cdot x) d\mu(c_{U_F}).$$

When χ is locally constant, this quantity relates to special values of partial L-functions in the following way.

Proposition 6.9. *Let* $\chi\colon \mathfrak{a}_p^{-1} - p\mathfrak{a}_p^{-1} \twoheadrightarrow \left(\mathfrak{a}^{-1} - p\mathfrak{a}^{-1}\right)/p^r\mathfrak{a}^{-1} \to \bar{\mathbb{Q}}$ *be an* $\mathcal{O}_F^\times$*-invariant function. Then,*

$$\int_{\mathbb{X}} \chi(c\tau^t \cdot x) d\mu(c_{U_F}) = \Delta_c(1_{[\mathfrak{a}],p}\chi, 0).$$

Proof. Consider the U_F-equivariant morphism

$$\varphi_{\chi,r}\colon \mathbb{D}(\mathbb{X}_r, \mathbb{Z}[1/m]) \longrightarrow \bar{\mathbb{Q}}_p,\ \lambda_r \longmapsto \sum_{\bar{x} \in \mathbb{X}_r} \chi(c\tau^t \cdot x)\lambda_r(\bar{x}).$$

Since χ factors through $\left(\mathfrak{a}^{-1} - p\mathfrak{a}^{-1}\right)/p^r\mathfrak{a}^{-1}$, it follows that

$$c_{U_F} \frown \varphi_\chi(\mu) = c_{U_F} \frown \varphi_{\chi,r}(\mu_r), \tag{25}$$

where $\mu_r \in H^{n-1}(\Gamma, \mathbb{D}(\mathbb{X}_r, \mathbb{Z}[1/m]))$ is the class described in Definition 4.3. In particular, $c_{U_F} \frown \varphi_{\chi,r}(\mu_r) \in \bar{\mathbb{Q}}$. Fix an embedding $\bar{\mathbb{Q}} \subset \mathbb{C}$. Then, the right-hand side of (25) can be computed using a representative of the image of μ_r in $H^{n-1}(\Gamma, \mathbb{D}(\mathbb{X}_r, \mathbb{R}))$. By Proposition 4.10, such a representative is given by

$$\varphi_r\colon \Gamma^n \longrightarrow \mathbb{D}(\mathbb{X}_r, \mathbb{R}),\ (\gamma_0, \ldots, \gamma_{n-1}) \longmapsto \left(\bar{x} \longmapsto \int_{\Delta(\gamma_0 z, \ldots, \gamma_{n-1} z)} (x/p^r)^* {}_c E_\psi\right),$$

where $z \in \mathscr{X}$ denotes an arbitrary point and $\Delta(\gamma_0 z, \ldots, \gamma_{n-1} z)$ is the geodesic simplex in $\mathscr{X}$ with vertices $\{\gamma_i z\}_i$. Hence, (25) can be written as

$$\int_{X(F)} \iota_\tau^* \sum_{\bar{x} \in \mathbb{X}_r} \chi(c\tau^t \cdot x)(x/p^r)^* {}_c E_\psi = \int_{X(F)} \iota_\tau^* E_{\tau,\chi},$$

where $X(F)$ is given in (21) and $E_{\tau,\chi}$ in (23). By Theorem 6.8, the result follows. □

Let $\bar{U}_F$ denote the completion of U_F in $\mathcal{O}_{F,p}^\times$. The previous proposition has an interpretation in terms of measures on $\mathfrak{a}_p^{-1} - p\mathfrak{a}_p^{-1}/\bar{U}_F$, that we proceed to explain. Cap product with c_{U_F} yields the morphism

$$H^{n-1}(U_F, \mathbb{D}(\mathbb{X}, \mathbb{Z}[1/m]) \longrightarrow \mathbb{D}(\mathbb{X}, \mathbb{Z}[1/m])_{U_F}.$$

In addition, if we identify $\mathbb{X} \xrightarrow{\sim} \mathfrak{a}_p^{-1}$ via dot product with $\tau \in F^n$, and denote $\pi\colon \mathfrak{a}_p^{-1} - p\mathfrak{a}_p^{-1} \twoheadrightarrow (\mathfrak{a}_p^{-1} - p\mathfrak{a}_p^{-1})/\bar{U}_F$ the quotient map, we can define

$$h\colon \mathbb{D}(\mathbb{X}, \mathbb{Z}[1/m])_{U_F} \longrightarrow \mathbb{D}\left((\mathfrak{a}_p^{-1} - p\mathfrak{a}_p^{-1})/\bar{U}_F, \mathbb{Z}[1/m]\right)$$

in the following way: for $[\mu] \in \mathbb{D}(\mathbb{X}, \mathbb{Z}[1/m])_{U_F}$ and $V \subset (\mathfrak{a}_p^{-1} - p\mathfrak{a}_p^{-1})/\bar{U}_F$ open set, $h([\lambda])(V) := \lambda(\pi^{-1}(V))$. We can then consider these two maps and view $c_{U_F} \frown \mu_{|U_F}$ as an element in $\mathbb{D}(\mathfrak{a}_p^{-1} - p\mathfrak{a}_p^{-1}/\bar{U}_F, \mathbb{Z}[1/m])$.

Moreover, the bijection given in (24) considered for every $r \geq 1$, induces a homeomorphism

$$\mathfrak{a}_p^{-1} - p\mathfrak{a}_p^{-1}/\bar{U}_F \xrightarrow{\sim} H_{\mathfrak{a}} \subset G = \varprojlim G_{p^r}.$$

Corollary 6.10. *Let $\mu_{\mathfrak{a}}$ be the measure on $H_{\mathfrak{a}}$ considered in Theorem 6.1. Via the homomorphism above, we have $c_{U_F} \frown \mu_{|U_F} = \mu_{\mathfrak{a}}$. In particular, for $s \in \mathbb{Z}_p$*

$$L_p(1_{[a],p}, s) = \int_{\mathbb{X}} \left\langle \mathrm{N}(\mathfrak{a})\mathrm{N}_{F_p/\mathbb{Q}_p}(c\tau^t \cdot x)\right\rangle^{-s} d\mu(c_{U_F}).$$

Proof. The equality $c_{U_F} \frown \mu_{|U_F} = \mu_{\mathfrak{a}}$ follows from the discussion above, Proposition 6.9 and Theorem 6.1. ☐

As a consequence, we obtain the relation between the local trace of $J_{E,\mathcal{L}}[\tau]$ and the local trace of the logarithm of a Gross–Stark unit.

Theorem 6.11. *Let $u \in \mathcal{O}_H[1/p]_-^\times \otimes \mathbb{Q}$ be the Gross–Stark unit introduced in Proposition 6.2. We have,*

$$\mathrm{Tr}_{F_p/\mathbb{Q}_p} J_{E,\mathcal{L}}[\tau] = \mathrm{Tr}_{F_p/\mathbb{Q}_p} \log_p(u^{\sigma_{\mathfrak{a}}}).$$

Proof. By viewing $\mathbb{D}_0(\mathbb{X}, \mathbb{Z}[1/m]) \subset \mathbb{D}(\mathbb{X}, \mathbb{Z}[1/m])$, we can consider $c_{U_F} \frown \varphi_{\chi_s}(\mu_0) \in \bar{\mathbb{Q}}_p$, where $s \in \mathbb{Z}_p$ and

$$\chi_s\colon \mathfrak{a}_p^{-1} - p\mathfrak{a}_p^{-1} \longrightarrow \bar{\mathbb{Q}}_p^\times,\ \alpha \longmapsto \langle \mathrm{N}(\mathfrak{a})\mathrm{N}_{F_p/\mathbb{Q}_p}(\alpha)\rangle^{-s}.$$

Moreover, since μ_0 is a lift of μ, Corollary 6.10 implies that for every $s \in \mathbb{Z}_p$,

$$c_{U_F} \frown \varphi_{\chi_s}(\mu_0) = L_p(1_{[\mathfrak{a}],p}, s).$$

Then, it follows from the definition of $J_{E,\mathcal{L}} = \mathrm{ST}(\mu_0)$, and the fact that μ_0 takes values on measures of total mass zero, that $\mathrm{Tr}_{F_p/\mathbb{Q}_p} J_{E,\mathcal{L}}[\tau] = -L_p'(1_{[\mathfrak{a}],p}, 0)$. Hence, the result follows from Theorem 6.4. ☐

7. Conjecture on the values of the log-rigid class

In this section, we make a conjecture on the values of the log-rigid class $J_{E,\mathcal{L}}$ at certain points $\tau \in X_p$ attached to totally real fields where p is inert. Then, we study the conjecture for the concrete case that $F/\mathbb{Q}$ is Galois and the point τ corresponds to a $\mathrm{Gal}\,(F/\mathbb{Q})$-stable ideal of F. Finally, we provide an observation that motivates the conjecture.

7.1. **Conjecture on the values of $J_{E,\mathcal{L}}[\tau]$.** Here and for the rest of the section, we consider the same notation as in Section 6. In particular, let F be a totally real field where p is inert, let $\mathfrak{a}$ be an integral ideal of F coprime to pc, and fix $\tau \in F^n$ a vector whose entries give an oriented $\mathbb{Z}$-basis of $\mathfrak{a}^{-1}$, which yields a point $\tau \in X_p$. We can then consider the value $J_{E,\mathcal{L}}[\tau] \in F_p$. Moreover, let H be the narrow Hilbert class field of F, $\mathfrak{p}$ the fixed prime ideal of H above p determined by the embedding $\bar{\mathbb{Q}} \subset \bar{\mathbb{Q}}_p$, and recall that we have the inclusion $H \subset H_{\mathfrak{p}} = F_p$. Also note that the p-adic logarithm can be extended to a map

$$\log_p \colon H_{\mathfrak{p}}^{\times} \otimes \mathbb{Q} \longrightarrow H_{\mathfrak{p}}$$

by linearity. In view of Theorem 6.11, we make the following conjecture.

Conjecture 7.1. *Suppose that F is a totally real field where p is inert. Let $\tau \in X_p$ be as above and let $u \in \mathcal{O}_H[1/p]_-^{\times} \otimes \mathbb{Q}$ be the Gross–Stark unit determined in Proposition 6.2. Then,*

$$J_{E,\mathcal{L}}[\tau] = \log_p(u^{\sigma_{\mathfrak{a}}}),$$

where $\sigma_{\mathfrak{a}} \in \mathrm{Gal}\,(H/F)$ denotes the Frobenius associated to the class of $\mathfrak{a}$.

When $n = 2$, the conjecture is true by Theorem B of [DPV24] since, as stated in Remark 5.9, we have the equality $J_{E,\mathcal{L}} = \log_p(\mathcal{J}_{\mathrm{DR}})$.

7.2. **The case of Galois extensions.** Suppose that F is Galois over $\mathbb{Q}$. If the narrow ideal class $[\mathfrak{a}]$ is $\mathrm{Gal}\,(F/\mathbb{Q})$-stable, we prove that $\log_p(\sigma_{\mathfrak{a}} u) \in \mathbb{Q}_p$. If moreover the ideal $\mathfrak{a}$ is $\mathrm{Gal}\,(F/\mathbb{Q})$-stable, we show that $J_{E,\mathcal{L}}[\tau] \in \mathbb{Q}_p$. Thus, Conjecture 7.1 follows from Theorem 6.11 in the case that $\mathfrak{a}$ is $\mathrm{Gal}\,(F/\mathbb{Q})$-stable.

Observe that under these assumptions, H is Galois over $\mathbb{Q}$. Denote by $D_{\mathfrak{p}} \subset \mathrm{Gal}\,(H/\mathbb{Q})$ the decomposition group at $\mathfrak{p}$. Note that $\mathrm{Gal}\,(H/\mathbb{Q})$, and therefore also $D_{\mathfrak{p}}$, act on $\mathcal{O}_H[1/p]_-^{\times} \otimes \mathbb{Q}$.

Lemma 7.2. *Let u be the Gross–Stark unit as above and let $[\mathfrak{a}]$ be a narrow ideal class that is $\mathrm{Gal}\,(F/\mathbb{Q})$-fixed. For every $\eta \in D_{\mathfrak{p}}$, we have $\eta(\sigma_{\mathfrak{a}} u) = \sigma_{\mathfrak{a}} u$ in $\mathcal{O}_H[1/p]_-^{\times} \otimes \mathbb{Q}$.*

Proof. We will use the uniqueness property determining Gross–Stark units given Proposition 6.2. For every ideal $\mathfrak{b}$ of $\mathcal{O}_F$, denote by $\sigma_{\mathfrak{b}} \in \mathrm{Gal}\,(H/F)$ the corresponding Frobenius and observe

$$\sigma_{\mathfrak{b}}\eta\sigma_{\mathfrak{a}}(u) = \eta\eta^{-1}\sigma_{\mathfrak{b}}\eta\sigma_{\mathfrak{a}}(u) = \eta\sigma_{\eta^{-1}(\mathfrak{b})}\sigma_{\mathfrak{a}}(u) = \eta\sigma_{\eta^{-1}(\mathfrak{b}\mathfrak{a})}(u),$$

where we used the $\mathrm{Gal}\,(F/\mathbb{Q})$-equivariance of the Artin map in the second equality, and the fact that $[\mathfrak{a}]$ is Galois fixed in the last one. From there,

$$\begin{aligned}
\mathrm{ord}_{\mathfrak{p}}(\sigma_{\mathfrak{b}}\eta\sigma_{\mathfrak{a}}(u)) &= \mathrm{ord}_{\mathfrak{p}}(\eta\sigma_{\eta^{-1}(\mathfrak{b}\mathfrak{a})}(u)) \\
&= \mathrm{ord}_{\mathfrak{p}}(\sigma_{\eta^{-1}(\mathfrak{b}\mathfrak{a})}(u)) \\
&= \Delta_c(1_{[\eta^{-1}(\mathfrak{b}\mathfrak{a})]}, 0) \\
&= \Delta_c(1_{[\mathfrak{b}\mathfrak{a}]}, 0),
\end{aligned}$$

where we used that $\tilde{\eta}(\mathfrak{p}) = \mathfrak{p}$ in the second equality, Proposition 6.2 in the second to last equality, and the last equality follows from $L(1_{[\eta^{-1}(\mathfrak{b}\mathfrak{a})]}, s) = L(1_{[\mathfrak{b}\mathfrak{a}]}, s)$, which can be verified from their definition. From the uniqueness asserted in Proposition 6.2, it can be deduced $\eta(\sigma_{\mathfrak{a}} u) = \sigma_{\mathfrak{a}} u$ in $\mathcal{O}_H[1/p]_-^{\times} \otimes \mathbb{Q}$ and we are done. ☐

Proposition 7.3. *Let $u \in \mathcal{O}_H[1/p]^\times_- \otimes \mathbb{Q}$ be the Gross–Stark unit introduced above and let $[\mathfrak{a}]$ be a narrow ideal class that is* $\mathrm{Gal}\,(F/\mathbb{Q})$*-fixed. We have $\log_p(\sigma_{\mathfrak{a}} u) \in \mathbb{Q}_p$.*

Proof. We will see that for every $\tilde{\eta} \in \mathrm{Gal}\,(H_{\mathfrak{p}}/\mathbb{Q}_p)$, we have $\tilde{\eta}\log_p(\sigma_{\mathfrak{a}} u) = \log_p(\sigma_{\mathfrak{a}} u)$. Consider the isomorphism given by extending an automorphism in $D_{\mathfrak{p}} \subset \mathrm{Gal}\,(H/\mathbb{Q})$ to $H_{\mathfrak{p}}$.

$$D_{\mathfrak{p}} \xrightarrow{\sim} \mathrm{Gal}\,(H_{\mathfrak{p}}/\mathbb{Q}_p),\ \eta \longmapsto \tilde{\eta}.$$

Observe that the map induced by the p-adic logarithm

$$\log_p \colon \mathcal{O}_H[1/p]^\times_- \otimes \mathbb{Q} \longrightarrow H_{\mathfrak{p}}^\times \otimes \mathbb{Q} \longrightarrow H_{\mathfrak{p}}$$

satisfies the following invariance property: for every $\eta \in D_{\mathfrak{p}}$ and $x \in \mathcal{O}_H[1/p]^\times_- \otimes \mathbb{Q}$, we have $\log_p(\eta x) = \tilde{\eta}\log_p(x)$. Indeed, this follows from the definition of $D_{\mathfrak{p}}$ and the $\mathrm{Gal}\,(H_{\mathfrak{p}}/\mathbb{Q}_p)$-invariance of the p-adic logarithm on $H_{\mathfrak{p}}^\times$. Applying this to $x = \sigma_{\mathfrak{a}}(u)$ and using Lemma 7.2 we obtain the desired result. ☐

Remark 7.4. Suppose that c satisfies the condition stated in Remark 6.3. Then, $u \in \mathcal{O}_H[1/p]^\times_-$ and its image under the embedding $u \in H^\times \subset H_{\mathfrak{p}}^\times = F_p^\times$ lands in $\mathbb{Q}_p$. Therefore, by Gross–Stark,

$$\log_p(u) = \frac{1}{n} L_p'(1_{[\mathcal{O}_F],p}, 0).$$

Since the valuation of u at $\mathfrak{p}$ is equal to $\Delta_c(1_{[\mathcal{O}_F]}, 0)$, we deduce that, up to a root of unity in $F_p^\times$ (see [DK24, Remark 2.7] for a discussion on this ambiguity),

$$u = p^{\Delta_c(1_{[\mathcal{O}_F]},0)} \exp\left(\frac{1}{n} L_p'(1_{[\mathcal{O}_F],p}, 0)\right).$$

This gives an explicit formula for u in terms of L-values, generalizing the type of abelian extensions of F that can be constructed only from p-adic L-functions, and in particular Proposition 3.14 of [Gro81] (see also Remark 7 of [DDP11]).

We proceed to study the invariant $J_{E,\mathcal{L}}[\tau]$ in the case that the ideal $\mathfrak{a}$ is $\mathrm{Gal}\,(F/\mathbb{Q})$-stable. In this setting, the isomorphism (19) induces an embedding

$$\mathcal{O}_F^\times \rtimes \mathrm{Gal}\,(F/\mathbb{Q}) \hookrightarrow \mathrm{GL}_n(\mathbb{Z})$$

determined by the following equations: for every $x \in \mathbb{Q}^n$, $\alpha \in F^\times$ and $\sigma \in \mathrm{Gal}\,(F/\mathbb{Q})$,

$$\alpha(\tau^t \cdot x) = \tau^t A_\alpha x,\ \sigma(\tau^t \cdot x) = \tau^t A_\sigma x.$$

Denote $\mathbb{D}_0 := \mathbb{D}_0(\mathbb{X}, \mathbb{Z}[1/m])$. Recall that $\mathrm{GL}_n(\mathbb{Z})$ acts on $\mathbb{D}_0$ as follows: for $g \in \mathrm{GL}_n(\mathbb{Z})$, $\lambda \in \mathbb{D}_0$, and $U \subset \mathbb{X}$ compact open

$$(g \cdot \lambda)(U) := \lambda(g^{-1}U).$$

Consider also the $\mathrm{GL}_n(\mathbb{Z})$-module $\mathbb{D}_0(\det) := \mathbb{D}_0 \otimes_{\mathbb{Z}[1/m]} \mathbb{Z}[1/m](\det)$. We use these actions and the embedding above to describe an action of $\mathcal{O}_F^\times$ and of $\mathrm{Gal}\,(F/\mathbb{Q})$, on $\mathbb{D}_0$ and $\mathbb{D}_0(\det)$. In particular, since $\{1\} \rtimes \mathrm{Gal}\,(F/\mathbb{Q})$ normalizes $U_F \rtimes \{1\}$, we have natural actions of $\mathrm{Gal}\,(F/\mathbb{Q})$ on $H^{n-1}(U_F, \mathbb{D}_0(\det))$ as well as on the coinvariants $(\mathbb{D}_0)_{U_F}$.

Recall the class $\mu_0 \in H^{n-1}(\Gamma, \mathbb{D}_0)_{\mathbb{Q}}$ given in (16). To lighten the notation for the next proof, by avoiding the appearance of tensor products, let $\ell \in \mathbb{Z}_{\geq 0}$ be such that $\ell\mu_0$ lifts to an element in $H^{n-1}(\Gamma, \mathbb{D}_0)$. Fix such a lift and denote it by $\tilde{\mu}_0 \in H^{n-1}(\Gamma, \mathbb{D}_0)$. Note that

$$\ell J_{E,\mathcal{L}}[\tau] = \int_{\mathbb{X}} \log_p(\tau^t \cdot x) d\lambda \in F_p,$$

where $\lambda := c_{U_F} \frown \tilde{\mu}_0 \in (\mathbb{D}_0)_{U_F}$ and $c_{U_F} \in H_{n-1}(U_F, \mathbb{Z})$ is the oriented fundamental class. This quantity is independent of the choice of lift $\tilde{\mu}_0$, as the difference between two such lifts is torsion.

Lemma 7.5. *Let $\tilde{\mu}_0 \in H^{n-1}(\Gamma, \mathbb{D}_0)$ and $c_{U_F} \in H_{n-1}(\Gamma, \mathbb{Z})$ be as above. The element $\lambda = c_{U_F} \frown \tilde{\mu}_0 \in (\mathbb{D}_0)_{U_F}$ is fixed by* $\mathrm{Gal}\,(F/\mathbb{Q})$.

Proof. By Shapiro's lemma, $\tilde{\mu}_0 \in H^{n-1}(\Gamma, \mathbb{D}_0)^-$ admits a unique lift via the isomorphism given by restriction

$$H^{n-1}(\mathrm{GL}_n(\mathbb{Z}), \mathbb{D}_0(\det)) \xrightarrow{\sim} H^{n-1}(\Gamma, \mathbb{D}_0)^-,$$

that we will also denote by $\tilde{\mu}_0$. It then follows that the restriction of $\tilde{\mu}_0$ to U_F is $\mathrm{Gal}\,(F/\mathbb{Q})$-invariant, as it can be obtained as the image of $\tilde{\mu}_0$ via the following maps

$$H^{n-1}(\mathrm{GL}_n(\mathbb{Z}), \mathbb{D}_0(\det)) \longrightarrow H^{n-1}(U_F \rtimes \mathrm{Gal}\,(F/\mathbb{Q}), \mathbb{D}_0(\det)) \longrightarrow H^{n-1}(U_F, \mathbb{D}_0(\det))^{\mathrm{Gal}\,(F/\mathbb{Q})}.$$

The result follows as cup product with c_{U_F} induces a $\mathrm{Gal}\,(F/\mathbb{Q})$-equivariant map

$$H^{n-1}(U_F, \mathbb{D}_0(\det)) \xrightarrow{\sim} (\mathbb{D}_0)_{U_F},$$

which can be verified via a direct calculation. □

Theorem 7.6. *Suppose that the coordinates of $\tau \in F^n$ given an oriented $\mathbb{Z}$-basis of a* $\mathrm{Gal}\,(F/\mathbb{Q})$*-stable ideal $\mathfrak{a}^{-1}$. We have, $J_{E,\mathcal{L}}[\tau] \in \mathbb{Q}_p$.*

Proof. We need to see that $J_{E,\mathcal{L}}[\tau] \in F_p$ is fixed by $\mathrm{Gal}\,(F_p/\mathbb{Q}_p)$. For every $\tilde{\sigma} \in \mathrm{Gal}\,(F_p/\mathbb{Q}_p)$ denote by $\sigma \in \mathrm{Gal}\,(F/\mathbb{Q})$ its restriction to F and note

$$\begin{aligned} \ell J_{E,\mathcal{L}}[\tau]^{\tilde{\sigma}} &= \int_{\mathbb{X}} \log_p(\sigma(\tau^t \cdot x)) d\lambda \\ &= \int_{\mathbb{X}} \log_p(\tau^t A_\sigma x) d\lambda \\ &= \int_{\mathbb{X}} \log_p(\tau^t \cdot x) d(A_\sigma \cdot \lambda) = \ell J_{E,\mathcal{L}}[\tau], \end{aligned}$$

where in the last equality we used that $\lambda \in (\mathbb{D}_0)_{U_\tau}$ is fixed by $\mathrm{Gal}\,(F/\mathbb{Q})$ by Lemma 7.5. □

Remark 7.7. Observe that in the theorem above, we only used that μ_0 is a group cohomology class for $\mathrm{SL}_n(\mathbb{Z})$ that belongs to the $w = -1$ eigenspace. Thus, the theorem can be applied to other rigid analytic classes.

Corollary 7.8. *Suppose that F is a totally real field that is Galois over $\mathbb{Q}$ and where p is inert. Let $\tau \in F^n$ with coordinates generating $\mathfrak{a}^{-1}$, where $\mathfrak{a}$ is a* $\mathrm{Gal}\,(F/\mathbb{Q})$*-stable ideal, and let $u \in \mathcal{O}[1/p]^\times_- \otimes \mathbb{Q}$ be the Gross–Stark unit of Proposition 6.2. We have,*

$$J_{E,\mathcal{L}}[\tau] = \log_p(u^{\sigma_{\mathfrak{a}}}).$$

7.3. **Example in the Galois case.** Consider the same notation as above. In this section, we give a numerical computation that exemplifies Proposition 7.3, namely the fact that the conjugates of Gross–Stark units corresponding to $\mathrm{Gal}\,(F/\mathbb{Q})$-fixed narrow ideal classes belong to $\mathbb{Q}_p$. We used the algorithm developed by Damm–Johnsen, see [Dam24], which is publicly available. We made minor modifications to the algorithm to output all conjugates of a given Gross–Stark unit.

Concretely, we take $p = 3$ and $F = \mathbb{Q}(\sqrt{D})$ with $D = 689$. The narrow Hilbert class field of F, denoted by H, is a cyclic extension of F of degree 8. The Galois group $\mathrm{Gal}\,(H/F)$ is isomorphic tothe narrow Hilbert class group of F, denoted by G_1. Denote by $\mathcal{F}_D$ the set of binary quadratic forms with integer coefficients and discriminant D. This set is equipped with a group action of $\mathrm{SL}_2(\mathbb{Z})$ by linear transformations and we have a bijection

$$\mathcal{F}_D/\mathrm{SL}_2(\mathbb{Z}) \xrightarrow{\sim} G_1$$

$$[Q] = [a, b, c] \longmapsto [\mathfrak{a}_Q] = \left[\left(a, \frac{-b+\sqrt{D}}{2}\right)\right],$$

where $[a, b, c]$ denotes the class of $ax^2 + bxy + cy^2$.

The Gross–Stark units are, up to high p-adic precision, roots of the polynomial

$$6561x^8 - 11340x^7 - 882x^6 + 4333x^5 + 2665x^4 + 4333x^3 - 882x^2 - 11340x + 6561.$$

More precisely, for every class $[Q] \in \mathcal{F}_D/\mathrm{SL}_2(\mathbb{Z})$, the table below gives the image of the Gross–Stark unit $\sigma_{\mathfrak{a}_Q}(u) \in H$ via the embedding $H \subset H_{\mathfrak{p}} = F_p$.

TABLE 1. $p = 3$, $D = 689$. Elements in $\mathcal{F}_D/\mathrm{SL}_2(\mathbb{Z})$ and their Gross–Stark unit.

$[Q]$	$\mathrm{ord}([\mathfrak{a}_Q])$	$\sigma_{\mathfrak{a}_Q}(u) \in F_p$
$[-20, 17, 5]$	8	$3^{-2}(7283498230698546457 + 204278114263245135 06\sqrt{D}) + O(3^{39})$
$[-10, 7, 16]$	2	$3^4 \cdot 2879930840163216397 + O(3^{45})$
$[-10, 17, 10]$	4	$25613292858296352193 + 34405602800800679412\sqrt{D} + O(3^{41})$
$[-5, 17, 20]$	8	$3^2(28389335835840796072 + 10412594344678893 69\sqrt{D}) + O(3^{43})$
$[5, 17, -20]$	8	$3^{-2}(7283498230698546457 + 16045184950846272897\sqrt{D} + O(3^{39})$
$[10, 7, -16]$	1	$3^{-4} \cdot 23094469614450736543 + O(3^{37})$
$[10, 17, -10]$	4	$25613292858296352193 + 20673935763701069 91\sqrt{D} + O(3^{41})$
$[20, 17, -5]$	8	$3^2(28389335835840796072 + 35431736942702897034\sqrt{D}) + O(3^{43})$

Note that $\sigma_{\mathfrak{a}_Q}(u) \in \mathbb{Q}_p$ if and only if $[\mathfrak{a}_Q]$ is 2-torsion in G_1. For real quadratic fields, this is equivalent to the fact that the class $[\mathfrak{a}_Q]$ is $\mathrm{Gal}\,(F/\mathbb{Q})$-fixed, as predicted by Proposition 7.3.

The work of Damm–Johnsen made it possible to verify this phenomenon in many additional cases. In these cases, the size of the narrow Hilbert class group ranged from 2 to 20.

7.4. **Further comments.** We conclude with some observations to support the conjecture for general $n \geq 2$. Denote $\mathbb{D} := \mathbb{D}(\mathbb{X}, \mathbb{Z}[1/m])$ and $\mathbb{D}_0 := \mathbb{D}_0(\mathbb{X}, \mathbb{Z}[1/m])$. Recall the class

$\mu \in H^{n-1}(\Gamma, \mathbb{D})^-$ constructed in Section 4 and denote by $\mu_{|U_F}$ its restriction to $U_F \subset \Gamma$. Consider the U_F-equivariant morphism

$$\bar{\mathcal{E}} \colon \mathbb{D} \longrightarrow F_p/\mathbb{Z}[1/m]\log_p(\mathcal{O}_F^\times),\ \lambda \longmapsto \int_{\mathbb{X}} \log_p(c\tau^t \cdot x) d\lambda,$$

where $\mathbb{Z}[1/m]\log_p(\mathcal{O}_F^\times)$ denotes the $\mathbb{Z}[1/m]$-span of $\log_p(\mathcal{O}_F^\times)$ in F_p. The proof of Theorem 6.11 implies that

$$c_{U_F} \frown \bar{\mathcal{E}}(\mu_{|U_F}) = \log_p(u^{\sigma_{\mathfrak{a}}}) \mod \mathbb{Z}_p \log_p(\mathcal{O}_F^\times). \tag{26}$$

Conjecture 7.1, predicts an expression for $\log_p(u^{\sigma_{\mathfrak{a}}})$ without the ambiguity $\mathbb{Z}_p\log_p(\mathcal{O}_F^\times)$. Observe that, if we consider measures of total mass zero, we can define the U_F-equivariant morphism

$$\mathcal{E} \colon \mathbb{D}_0 \longrightarrow F_p,\ \lambda \longmapsto \int_{\mathbb{X}} \log_p(c\tau^t \cdot x) d\lambda.$$

Moreover, it follows from Proposition 4.12 that $\mu_{|U_F}$ lifts to a class in $H^{n-1}(U_F, \mathbb{D}_0)$. However, the lift is not unique. Indeed, the long exact sequence

$$\cdots \longrightarrow H^{n-2}(U_F, \mathbb{Z}[1/m]) \xrightarrow{\delta} H^{n-1}(U_F, \mathbb{D}_0) \longrightarrow H^{n-1}(U_F, \mathbb{D}) \longrightarrow \cdots$$

shows that a lift of $\mu_{|U_F}$ is well-defined up to the image of δ. Since $U_F \simeq \mathbb{Z}^{n-1}$ by Dirichlet's unit theorem, we have a natural isomorphism

$$H^{n-2}(U_F, \mathbb{Z}[1/m]) \simeq H_1(U_F, \mathbb{Z}[1/m]) \simeq U_F \otimes \mathbb{Z}[1/m]. \tag{27}$$

This leads to the following proposition.

Proposition 7.9. *The map*

$$H^{n-2}(U_F, \mathbb{Z}[1/m]) \longrightarrow F_p,\ \varepsilon \longmapsto c_{U_F} \frown \mathcal{E}(\delta(\varepsilon))$$

has image equal to $\mathbb{Z}[1/m]\log_p(U_F)$. *More precisely, via the natural isomorphism given in* (27), *it is equal to* $\log_p \colon U_F \otimes \mathbb{Z}[1/m] \xrightarrow{\sim} \mathbb{Z}[1/m]\log_p(U_F)$.

In other words, the process of lifting $\mu_{|U_F}$ to a class valued in total mass zero measures allows to compute its image under $\mathcal{E}$ and construct an element in F_p. However, since the lift is only well-defined up to $U_F \otimes \mathbb{Z}[1/m]$, the elements we construct in F_p are only well-defined up to $\mathbb{Z}[1/m]\log_p(U_F)$. Thus, we obtain a similar ambiguity for the Gross–Stark unit as the one appearing on the formula of the Gross–Stark conjecture.

However, in this paper, we worked with cohomology classes for Γ, instead of for U_F, to define our invariants. In this way, we obtained that $\mu \in H^{n-1}(\Gamma, \mathbb{D})^-_{\mathbb{Q}}$ has a unique lift $\mu_0 \in H^{n-1}(\Gamma, \mathbb{D}_0)^-_{\mathbb{Q}}$. Indeed, as explained in Section 4.4, this follows from the long exact sequence

$$H^{n-2}(\Gamma, \mathbb{Z}[1/m])^- \longrightarrow H^{n-1}(\Gamma, \mathbb{D}_0)^- \longrightarrow H^{n-1}(\Gamma, \mathbb{D})^- \longrightarrow H^{n-1}(\Gamma, \mathbb{Z}[1/m])^-,$$

Proposition 4.12, and the fact that $H^{n-2}(\Gamma, \mathbb{Z}_p)^-$ is torsion by [LS19]. Then, the restriction $\mu_{0|U_F}$ is a preferred lift of $\mu_{|U_F}$ to $H^{n-1}(U_F, \mathbb{D}_0(\mathbb{X}, \mathbb{Z}[1/m]))$. Hence, using $\mu_{0|U_F}$ and the map $\mathcal{E}$, we are able to produce a canonical element in F_p

$$c_{U_F} \frown \mathcal{E}(\mu_{0|U_F}) = J_{E,\mathcal{L}}[\tau] \in F_p.$$

The fact that this construction is unique suggests that the quantity we produced could be a preferred lift of $\mathrm{Tr}_{F_p/\mathbb{Q}_p} \log_p(u^{\sigma_{\mathfrak{a}}})$, and this motivates us to state Conjecture 7.1 above.

We summarize this discussion with the following commutative diagram

$$\begin{array}{ccccc}
\text{torsion} & \longrightarrow & H^{n-1}(\Gamma, \mathbb{D}_0)^- & \longrightarrow & H^{n-1}(\Gamma, \mathbb{D})^- \\
\downarrow & & \downarrow & & \downarrow \\
U_F \otimes \mathbb{Z}[1/m] & \xrightarrow{\delta} & H^{n-1}(U_F, \mathbb{D}_0) & \longrightarrow & H^{n-1}(U_F, \mathbb{D}) \\
\downarrow{\scriptstyle c_{U_F} \frown \mathcal{E}\circ\delta(\cdot)} & & \downarrow{\scriptstyle c_{U_F} \frown \mathcal{E}(\cdot)} & & \downarrow{\scriptstyle c_{U_F} \frown \bar{\mathcal{E}}(\cdot)} \\
\mathbb{Z}[1/m] \log_p(U_F) & \longrightarrow & F_p & \longrightarrow & F_p/(\mathbb{Z}[1/m] \log(U_F)).
\end{array}$$

References

[Ban+24] Kenichi Bannai et al. The Hodge realization of the polylogarithm and the Shintani generating class for totally real fields. *Adv. Math.* 448 (June 2024), p. 109716.

[BKL18] A. Beilinson, G. Kings, and A. Levin. Topological Polylogarithms and p-adic Interpolation of L-values of Totally Real Fields. *Math. Ann.* 371.3 (2018), pp. 1449–1495.

[BCG20] N. Bergeron, P. Charollois, and L. E. Garcia. Transgressions of the Euler class and Eisenstein cohomology of $\mathrm{GL}_N(\mathbf{Z})$. *Jpn. J. Math.* 15.2 (2020), pp. 311–379.

[BCG23] N. Bergeron, P. Charollois, and L. E. García. Eisenstein cohomology classes for GL_N over imaginary quadratic fields. *J. Reine Angew. Math.* 2023.797 (2023), pp. 1–40.

[BC92] J.-M. Bismut and J. Cheeger. Transgressed Euler classes of $\mathrm{SL}(2n, \mathbf{Z})$ vector bundles, adiabatic limits of eta invariants and special values of L-functions. *Ann. Sci. École Norm. Sup. (4)* 25.4 (1992), pp. 335–391.

[BS73] A. Borel and J-P. Serre. Corners and arithmetic groups. *Comment. Math. Helv.* 48 (1973), pp. 436–491.

[Bro82] K. S. Brown. *Cohomology of Groups.* Vol. 87. Graduate Texts in Mathematics. New York: Springer, 1982.

[Cha07] H. Chapdelaine. "Elliptic units in ray class fields of real quadratic number fields". PhD thesis. McGill University, Apr. 2007.

[Cha09] H. Chapdelaine. p-units in ray class fields of real quadratic number fields. *Compos. Math.* 145.2 (2009), pp. 364–392.

[CD14] P. Charollois and S. Dasgupta. Integral Eisenstein cocycles on $\mathbf{GL}_n$, I: Sczech's cocycle and p-adic L-functions of totally real fields. *Camb. J. Math.* 2.1 (2014), pp. 49–90.

[CDG15] P. Charollois, S. Dasgupta, and M. Greenberg. Integral Eisenstein cocycles on GL_n, II: Shintani's method. *Comment. Math. Helv.* 90.2 (2015), pp. 443–488.

[Dam24] Håvard Damm-Johnsen. "Modular algorithms for Gross-Stark units and Stark-Heegner points". *LuCaNT: LMFDB, computation, and number theory.* Vol. 796. Contemp. Math. Amer. Math. Soc., [Providence], RI, [2024] ©2024, pp. 261–284.

[DD] H. Darmon and S. Dasgupta. Elliptic Units for Real Quadratic Fields. *Ann. of Math.* 163 (2006), pp. 301–46.

[DDP11] H. Darmon, S. Dasgupta, and R. Pollack. Hilbert modular forms and the Gross-Stark conjecture. *Ann. of Math.* 174 (2011), pp. 439–484.

[DGL25] H. Darmon, L. Gehrmann, and M. Lipnowski. Rigid meromorphic cocycles for orthogonal groups. *Forum of Mathematics, Sigma* 13 (2025), e160. DOI: 10.1017/fms.2025.10035.

[DPV24] H. Darmon, A. Pozzi, and J. Vonk. The values of the Dedekind-Rademacher cocycle at real multiplication points. *J. Eur. Math. Soc.* 26 (2024), pp. 3987–4032.

[DV21] H. Darmon and J. Vonk. Singular moduli for real quadratic fields: a rigid analytic approach. *Duke Math. J.* 170.1 (2021), pp. 23–93.

[Das08] S. Dasgupta. Shintani zeta functions and Gross–Stark units for totally real fields. *Duke Math. J.* 143 (2008), pp. 225–279.

[DK23] S. Dasgupta and M. Kakde. On the Brumer–Stark conjecture. *Ann. of Math.* 197 (2023), pp. 289–388.

[DK24] S. Dasgupta and M. Kakde. Brumer–Stark units and explicit class field theory. *Duke Math. J.* 173.8 (2024), pp. 1477–1555.

[DS18] S. Dasgupta and M. Spieß. Partial zeta values, Gross's tower of fields conjecture, and Gross–Stark units. *J. Eur. Math. Soc.* 20.11 (2018), pp. 2643–2683.

[Das+23] S. Dasgupta et al. *The Brumer–Stark Conjecture over* $\mathbb{Z}$. Preprint. 2023. arXiv: `2310.16399 [math.NT]`. URL: `https://arxiv.org/abs/2310.16399`.

[DR80] P. Deligne and K. A. Ribet. Values of abelian L-functions at negative intgers over totally real fields. *Invent. Math.* 59 (1980), pp. 227–286.

[GS24] A. Galanakis and M. Spieß. Adelic Eisenstein classes and divisibility properties of Stickelberger elements. (2024). arXiv: `2402.11583 [math.NT]`. URL: `https://arxiv.org/abs/2402.11583`.

[GGM25] L. Gehrmann, X. Guitart, and M. Masdeu. *Evaluation of Bianchi rigid meromorphic cocycles at big ATR points.* Preprint. 2025.

[Gro81] B. H. Gross. p-adic L-series at $s = 0$. *Journal of the Faculty of Science, University of Tokyo, Section IA Mathematics* 28 (1981), pp. 979–994.

[LS19] J.-S. Li and B. Sun. Low degree cohomologies of congruence groups. *Sci. China Math.* 62 (2019), pp. 2287–2308.

[MQ86] V. Mathai and D. Quillen. Superconnections, Thom classes, and equivariant differential forms. *Topology* 25 (1 1986), pp. 85–110.

[MS74] J. W. Milnor and J. D. Stasheff. *Characteristic Classes.* Vol. 76. Ann. of Math. Studies. Princeton, NJ: Princeton University Press, 1974, p. 330.

[Put82] M. van der Put. Les fonctions theta d'une courbe de Mumford. *Groupe de travail d'analyse ultrametrique* 9 no. 1, Talk no. 10, 12 (1981-1982).

[SS91] P. Schneider and U. Stuhler. The cohomology of p-adic symmetric spaces. *Invent. Math.* 105 (1991), pp. 47–122.

[Scz93] R. Sczech. Eisenstein group cocycles for GL_n and values of L-functions. *Invent. Math.* 113.3 (1993), pp. 581–616. URL: `https://eudml.org/doc/144138`.

[Ser71] J-P. Serre. Cohomologie des groupes discrets. *Ann. of Math. Studies* 70 (1971), pp. 77–169.

[Sie79] C. L. Siegel. Uber die Fourierschen Koeffizienten von Modulformen. *Gesammelte Abhandlungen* (1979), pp. 98–139.

[Spi14] Michael Spiess. Shintani cocycles and the order of vanishing of p-adic Hecke L-series at $s = 0$. *Math. Ann.* 359.1-2 (2014), pp. 239–265.

[Ven15] K. Ventullo. On the rank one abelian Gross-Stark conjecture. *Comment. Math. Helv.* 90 (2015), pp. 939–963.

[Xu23] P. Xu. "Arithmetic Eisenstein Theta Lifts". PhD thesis. Montreal, Canada: McGill University, Apr. 2023.

[Xu25] Peter Xu. *Symbols for toric Eisenstein cocycles and arithmetic applications.* Preprint. 2025.

M. R.: IMJ-PRG, Sorbonne Université, Paris, France

Email address: `marti.roset-julia@imj-prg.fr`

P. X.: UCLA, Los Angeles, USA

Email address: `peterx@math.ucla.edu`